PROFESSOR: So we've been developing the procedure to describe a quantum gas of identical particles. That's our non-interacting. So what does that mean? Is that we are thinking about the Hamiltonian for a system of N particles. If they are not interacting, it can be written as the sum of the contributions for the individual particles. And if they are identical, it's basically the same particle for each one of them. Same Hamiltonian for each one of them.

OK? How do we handle these? Well, if you regard this as a quantum problem, for a single particle-- so for each one of these Hamiltonians-- we can find the eigenstates. So there's a bunch of states that say label by k because we are ultimately have in mind the plane waves that describe gas particles in a box. And there is corresponding energy levels for the single particles, epsilon k. And the single particle spectrum can therefore can be described by a bunch of energies, a lot of energy levels, with different values of k and different allowed discretized energies.

Now, when we want to describe a case of N particles that are non-interacting, we said that we could, in principle, pick one k here, one k here, one k here. And if we are dealing with bosons, potentially more than one at each side. And so the state would then be described by a bunch of occupation numbers Nk. And this Nk will be 0 or 1 if you are talking about fermions, but any integer if you are talking about bosons. So this eta being plus or minus was used to distinguish appropriate symmetrization or anti-symmetrization.

Now, once we have a state such as this, clearly the N particle hamintonium acting on a state that is symmetrized with these bunch of occupation numbers will give us back this state. The energy that we would be getting is summing over all possible k values, epsilon k times the number of occurrance of that in this state. Again,

multiplying this state back. Of course, we have a constraint that the number of particles matches the sum of the occupation numbers.

So then we said, OK, this is just a description of the state that they want to describe, or the set of states that represent this system of identical non-interacting partners in quantum mechanics. So the next stage, we want to do some kind of first statistics for them. And we had seen that basically, in order to assign appropriate statistics in case of quantum systems, we have to deal with a density matrix. And we saw that in the canonical representation, the density matrix is complicated and amounts to some kind of interaction among particles. Although there is no interaction at this level, the symmetrization or anti-symmetrization create states that are intermixed in a manner that appears as if you have an interaction.

We found that, however, we could get rid of that by going to the grand canonical ensemble where we would trade in the constraint that the number of particles is fixed with a chemical potential. And in this ensemble, we found that the density matrix is diagonal in this representation. So the diagonal elements can be thought of as the probability of one of these occupation number sets. And so the probability of a particular set of occupations in this grand canonical prescription is proportional to e to the beta-- of course we have mu if we are going to this grand canonical ensemble-- minus epsilon k, and then we had Nk. So the probability that a particular one of these is occupied Nk times in this grand canonical prescription is simply an exponential.

And of course, the different k levels are completely independent of each other. So the probability factor arises in this fashion. Of course, as any probability, we have to normalize this. Normalization is appropriate. We call the grand canonical partition function and will be different whether we are dealing with boson statistics or fermion statistics.

Actually, because we have a structure such as this, Q is going to be the product of the normalization of each individual one. And therefore, log of Q is going to be a sum of the contributions that I would get for the normalization of each individual

ones. And of course, the log of whatever normalization I have for the individual one.

If I have the case of a fermions where Nk gets 0 or 1 as the two possibilities, then the normalization will be simply 1 plus this exponential. And I choose to write that as z e to the minus beta epsilon of k where I have introduced z to be e to the beta mu. Because this combination occurs all the time, I don't want to keep repeating that.

So this was for the case of fermions. For the case of bosons where N goes 0, 1, 2, 3, et cetera, this is a geometric series. For the geometric series, when I sum it, I would get 1 over this factor. So when I take the log I will have a minus sign here. And I can put the Boson and Fermi cases together by simply putting a factor of minus eta here so that I have the plus for the case of fermions and the minus for the case of bosons. OK?

So this is one thing that we will be using. The other thing is that this is the probability for this level having a certain occupation number. In k we can always ask, what is the average of Nk? And from this probability it is easy to calculate the average. And the answer, again, can be collapsed into one expression for the case of bosons and fermions as 1 over z inverse e to the beta epsilon of k minus eta encompassing both fermions and bosons.

So these results give some idea both at the microscopic level-- in terms of occupation numbers of these different levels-- and macroscopic level a grand partition function from which we can construct thermodynamics. Actually, full probability, again, in this ensemble which we can transform in principle and get density matrices, et cetera, in other basis. But these are the kinds of things that we can do for any interacting Hamiltonian. OK?

Now, what I said over here was that I'm actually interested in the case of a gas. So let's again think about the type of gas such as the one in this room that we are interested. And again, to make a contrast with something that we may want to do later, let's emphasize that it is a non-relativistic gas.

So the Hamiltonian H is essentially free particle in a box of volume V, the thing that

we have been using all the time, just the kinetic energy of a one particle state. And for that we know that the single particle energy levels are characterized by a vector k. As three components, in fact. And h bar squared over 2mk squared. And we've discussed what the discretization of k is appropriate to the size of the box. We won't go over that.

So again, maybe it is important to mention that because of the discretizatoin of the k values, the spacing between possible values of k are 2 pi over l in each direction. And ultimately what we want to do is to replace these sums over k's in the limit of large boxes with integrals over k. So these go over to an integral over k times the density factor that you get in going from the sum to the integral, which is 2 pi cubed times the volume.

I will introduce one other factor here, which is that when we are talking about quantum particles and bosons and fermions, they are typically assigned also another state-- another parameter-- that characterizes their spin. And depending on the possible values of spin, you will have g copies of the system. And this g will be related to the quantum spin-- to the formula 2s 1-- so that when you have fermions, typically s is half integer and this will be an even number. Whereas for bosons it will be an odd number. So that's, again, something that we've seen. You just, essentially says multiply everything by some factor of g.

Now if I go through this procedure that I have outlined here, I will ultimately get the grand partition function of the gas-- which as we will see is related to its pressure-- in terms of the chemical potential. Throughout the course we've seen that the more interesting thing to look at is how things depend on density rather than chemical potential. So I'd better use this result to get what the density is.

How do I do that? Well, once I have determined what the chemical potential is, the average number of particles will be the sum over and the average occupation numbers that I have in all of these states. And according to the prescription that we have over here, in the limit of a large box, this sum over k becomes gV intergral d cubed k 2 pi Q. Ultimately I will divide by V so I will have the formula for the density.

And what is Nk? Well, I have the formula for Nk. It is 1 over z inverse e to the beta. Epsilon k is h bar squared k squared over 2m, and then I have a minus 8. OK? So if I divide through by V, really I will have the density. Get rid of this V. But I have to do that integral.

Now for this and a lot of other things, you can see that always I will have sums or integrations over the combination that appears in the exponent and enhances dimension list which is beta epsilon of k. So let me introduce a new variable. Let me call x to be the combination beta h bar squared k squared over 2m. So I'm going to call this e to the x, if you like. So I have changed variables from k to x. k in terms of x is simply square root of 2m inverse of beta-- which is kt-- divided by h bar squared. But I took a square root so it's just h bar. Times x to the 1/2. OK?

Now, I realize that h bar is in fact h divided by 2pi. So the 2pi I can put in the numerator. And then I remember that actually the combination h divided by square root of 2pi mkt, I was calling these thermal wavelength lambda. And k should have the dimensions of an inverse landscape, so that's the appropriate thing to choose.

So what do I get? I take a square root of pi inside and I'm left with 2 square root of pi divided by this parameter lambda x to the 1/2. Again, for later clarity, let's make sure we remember this was our lambda. So this is the variable k that I need.

In fact, for the-- for the integration I have to also make a change in dk. So I not that dk, if I take a derivative here, I will get x-- 1/2x to the minus 1/2 becomes root pi over lambda x to the minus 1/2 dx. Actually, the factor that I need over here is dk-- d cubed k over 2pi cubed.

Now everything only depends on the magnitude of k. So I will take advantage of spherical symmetry, write d cubed k as 4pi k squared dk divided by 8pi cubed. And then I substitute from k squared and dk above in terms of this x. So I will have a 1 over 2pi squared from this division. k squared is simply 4pi over lambda squared x. This is k squared. dk is root pi over lambda x to the minus 1/2 dx.

So you can see lots of things cancel out and the answer is going to be x to the 1/2

dx divided by lambda cubed. And what is left from all of these pi's et cetera, is a factor of square root of pi over 2. Or 2 in the numerator or a square root of pi over 2 in the denominator. OK?

So if I do these set of changes that I have on the right part of the board, what do I have for density? The density becomes simply g. I have this integral over k that I have recast as an integral that is x to the 1/2 dx. Let me pull out the factor of lambda cubed on the outside. So I have g over lambda cubed. I will pull out to that factor of root pi over 2 that I had before. And then I have the integral from 0 to infinity dx x to the 1/2.

Actually, let me write it in this fashion. dx x to the 1/2. And in the denominator, what do I have? I have z universe into the x minus 8. So we did all of these and we ended up with an integral that we don't recognize. So what we are going to do is to pretend that we recognize it and give it a name. All right?

So we define a set of functions f sub m of variable z. Once I do the integration over x, I will get a function of z. So that function of z I will call fm of z. But clearly I have a different fm if I have eta minus 1 or eta plus 1 for bosons of fermions. So there is two classes of things.

What's this m? OK. So here I had the integral x to the 1/2. It turns out that you will get a whole bunch of these integrals where the [INAUDIBLE] that is out front in the numerator is different. So I will generalize this x to the 1/2, this 1/2 to something that I'll call it m minus 1. And then I have z inverse e to the x minus eta. And if these normalized so that what is in front is 1 over m minus 1 factorial. OK? So these are very important functions. Everything we will do later on we will be expressing in terms of these functions.

Now, clearly what is happening here corresponds to m minus 1 being equal to 1/2 or m being 3/2. What is this normalization? Well, for the normalization, if this is 1/2 I would need 1/2 factorial. And if you go and look at the various definitions that we have for the gamma function or the factorial, you can check that 1/2 factorial is root pi over 2. And, indeed, we saw this already when we wrote the expression for the

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surface area in general dimensions that involved factorial function appearing as fractions. And in order to get the right result in two or three dimensions, you can check that this root pi over 2 for 1/2 factorial is correct.

So, indeed, I already have this root pi over 2 over here. It was properly normalized. So the whole thing here can be written as n is g over lambda cubed f 3/2 eta of z. That is the expression that relates the density to the chemical potential or this communication e to the beta mu, which is z or the fugacity is simply this nice formula. OK?

But what we want is to calculate something like the pressure for the gas. Well, let me remind you that log Q, essentially, quite generally, log Q comes from normalizing something that is like mu times the total number of particles minus the energy-- you can kind of read that from here-- and then there would be TS from the number of states. This was the general formula that we had.

And when we substitute for a system that is extensive, where E for a gas is mu n plus TS minus PV, we see that the log of the partition function is simply the combination beta PV. Of course, again, this pressure as a function of density or chemical potential would be different whether we are thinking about bosons of fermions.

But it can be calculated from the kind of sum that we have over here. So let's do that. So I have beta P which is something like log Q divided by V. Is obtained by doing this sum over k over V. And so what do I get? I will get-- first of all, I have the minus eta sum over k [INAUDIBLE] with g integral over k divided by 2pi cubed. The volume factor cancels because I divided by volume. I have log of 1 minus eta z e to the minus beta-- again, h bar squared k squared over 2m-- are the energy that I'm dealing with. OK?

Fine. What do we do with this? We make the same change of variables that we have over here from k to x. So then I would have minus eta d cubed k over 2pi cubed gets replaced with this 2 over root pi lambda cubed. And then I have the integral 0 to infinity dx x to the 1/2 log of 1 minus eta z e to the minus x. Huh. It

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seems like-- yes?

AUDIENCE: What happened to your factor g?

PROFESSOR: G? Good question. Anything else? So I was hoping to have integrals that are all expressible in terms of this-- these set of functions that I defined. This log does not fall into this category. Should I define something else? Well, the answer is no because I can make that integral look like this by simbly doing an integration by parts.

So we do integrate by parts in which case what I do is I take a derivative of this and I take an integral of x to the 1/2. Integral of x to the 1/2 becomes x to the 3/2 over 3/2. There is, of course, a minus sign involved. So this minus becomes plus eta. Make sure I have the g this time. 2 over root pi lambda cubed.

I have the integral from 0 to infinity dx. x to the 1/2 became x to the 3/2. The 3/2 I will bring out front. So I have 1 over 3/2 here. And then I have to take the derivative of this function. What's the derivative of the log? It is 1 over the argument of the log times the derivative of the argument of the log which is minus eta z, the derivative of e to the minus x, which is e to the minus x. And this minus becomes plus. OK?

So let's arrange things a little bit. I can get rid of these etas because eta squared will be 1. What is the answer, therefore? It is g over lambda cubed. That's fine. If root pi over 2 is 1/2 factorial, and I multiply 1/2 factorial with 3/2, what I will get is 1 over 3/2 factorial. Because n factorial is n times n minus 1 factorial.

And then, what do I have? I have 0 to infinity dx. x to the 3/2, divide the numerator and denominator by z e to the minus x and you get z inverse e to the x minus eta. And lo and behold, we have a function of the variety that we had defined on the board corresponding to n minus 1 being 3/2 or n being 5/2. So what we have here is that beta P, depending on whatever statistics we are dealing with, is g over lambda cubed f 5/2 eta of z. Yes?

AUDIENCE: [INAUDIBLE]

PROFESSOR: OK. So what you have to do is always ensure what is happening to the boundary terms. Boundary terms will be a product of x to the 3/2 and this factor. At x equals to 0, x to the 3/s will ensure that you are 0. At x goes to infinity, the exponential will ensure that you're to 0. So the boundary terms are important and indeed you have to check them. And you can check that they are 0. OK?

AUDIENCE: I think the factor that multiples x to the 3/2 is [INAUDIBLE]. But it's-- the argument still holds.

PROFESSOR: The argument still holds because as x goes to infinity, e to the minus x goes to 0. Log of 1 is 0. In fact, you can ask, how does it go to 0? It goes to 0 exponentially. Exponentially it will [INAUDIBLE]. Yes. Thank you for the correction. OK?

All right. There's one other thing that we can take a look at. The total energy. OK. So the energy in each one of these states is given by that formula. So if I ask, what is the total expectation value of energy? And again, in thermodynamics we expect the extensive quantities such as the energy should have, in an aerodynamic sense, well defined values so I won't write the expectation value. Is going to be a sum over k Nk epsilon k. And this will depend on the statistics that I'm looking at.

And what I will have is that this will go over to an integral over k divided by 2pi cubed multiplied g and multiplied by V. I have epsilon of k which is h bar squared k squared over 2m. The expectation value of N, which is z inverse e to the minus beta-- sorry-- z inverse e to the plus beta h bar squared k squared over 2m minus 8. OK?

So the simplest way to do this is to first of all divide by V. So I can define an energy density. Actually, since I'm going to change variables to this combination that is x, I notice that up to a factor of betas the numerator is x. So let me actually calculate the combination beta e over V. That only depends on this combination x.

So what is it? It is g-- the integral over k, I know how to replace it with something that involves x-- is going to give me 2 over root pi lambda cubed. Integral 0 to infinity dx. From the change of variables, I have x to the 1/2. But I introduced one

factor of x here because I have this energy factor in the numerator. So that changes this to x to the 3/2. And then again, I have z inverse e to the x minus 8. OK?

So if is one of the integrals that we have defined. In fact, it is exactly what I have above for the pressure, except that I don't have the factor of 3/2 that appeared nicely to give the right normalization. So what I conclude is that this quantity is in fact g over lambda cubed times 3/2 of f 5/2 eta of z. OK?

But they are exactly the same functions for energy and pressure. So you can see that, irrespective of what the chemical potential is doing, I can state that for the quantum system e is always 3/2 PV. Of course, we have seen this expression classically. Now what happens quantum mechanically is that the pressure starts to vary in some complicated way with density.

But what we learn is that energy will also vary in the same complicated way with density, such that always this formula EV 3/2 PV is valid. There is some nice symmetry reason for why that is the case that you explore in one of the problems that you will see. OK?

So these are the characteristics. In some sense we have solved the problem because we know both density and pressure in terms of chemical potential. So all we need to do is to invert this, get the chemical potential as a function of density, insert it in the formula for pressure and we have pressure as a function of density. But since these are kind of implicit through these f functions, we don't quite know what that means.

So the next stage is to really study how these f functions behave and how we can relate their behaviors to dependence on density. And we will see that we have to distinguish between low density and high density regimes. What does it mean? Well, you can see that both have players on the right hand side is dimensionless. The only dimension full combination that we have here is N-- well, we can make it dimensionless by multiplying by lambda cubed over g.

So it turns out that whether or not you see things that are classical or things that are

very quantum mechanical depend on this degeneracy factor. Which, within our context of statistical physics of gases, we define to be the communication N lambda cubed over g. And we will regain classical behaviors when degeneracy is small, density is small enough such that d is much less than 1. We will see totally new behavior that is appropriate to quantum mechanics in other limit where [? d ?] becomes large. OK?

So this distinguishes classical versus quantum behavior. Now, we expect to regain classical behavior in the limit of very low densities. That's what we've been kind of saying from the very first lecture that low density gas has this ideal behavior where PV is Nkt. And, therefore, let's focus on the limit where V is much less than 1, which is classical. And since d is f 3/2 eta of z, we need the behavior of the function when it is small-- and I'll show you [INAUDIBLE] consistently that the function is small when it's argument is small. So we'll see that d less than 1 implies or is implied by z being less than 1. OK?

So let's take a look at one of these functions, fm eta z, 1 over m minus 1 factorial, 0 to infinity dx, x to the m minus 1, z universe e to the x minus 1. And I'd like to explore the limit of this when z is small. So that's currently not very good because I have z inverse. So what I will do is I will multiply by z e to the minus x both numerator and denominator. So I will have m minus 1 factorial integral 0 to infinity dx x to the n minus 1. I have z e to the minus x. I have-- oops, there was an eta here. 1 minus eta z e to the minus x. OK?

Then I say, well, the thing that is appearing in the denominator, I can write as a sum of a geometric series in powers of z. And that's, after all, what I'm attempting to do, make an expansion in powers of z. It starts with z to the minus x. And 1 over 1 minus eta z to the minus x I can write as the sum alpha going from 0 to infinity of eta z e to the minus x raised to the power of [INAUDIBLE]. This geometric series sums to what I have over there. OK?

Let's rearrange this a little bit. I will take the sum over alpha outside. Well, I notice that actually I have here, this first term, which is different from this by a factor of eta.

So let me multiply by an eta and get another eta. And then I have a sum that starts from 1 to infinity of eta z e to the minus x to the power of alpha. So I have eta z to the power of alpha. I have one other eta here. And then I have the integration over x, x to the m minus 1. I didn't write e to the minus x to the alpha because I pulled it out the integration. I will write it now. e to the minus alpha x. And I didn't write 1 over n minus 1 factor.

Now, once more, this is an integration that you have seen. The result of doing integral dx x to the m minus 1 e to the minus x is simply m minus 1 factorial. Except that it is not e to the minus x but it's e to the minus alpha x. I can rescale everything by alpha and I will pull out a factor of alpha to there. OK?

So what do we have? We have that fm eta of z as a [? power series ?] representation, alpha running from 1 to infinity, z to the alpha divided by alpha to the m. It's an alternating series that starts with plus and then becomes a minus, et cetera. So if I were to explicitly write this, you will see how nice it is. It is z plus eta z squared 2 to the m plus z cubed 3 to the m plus eta z to the 4 4 to the m. And you can see why it is such a nice thing that really deserves to have a function named after it. So we call that nice series [? to Vf. ?]

Now, for future reference, note the following. If I take a derivative of this function with respect to z-- its argument-- then in this power series, say, z to the 4 becomes 4z cubed. 4 will cancel one of the 4's that I have here and it becomes 4 to the m minus 1. This becomes 3 to the m minus 1. So you can see that the coefficients that were powers of m become powers of m minus 1.

Numerator has gone down by one factor. So let's multiply by z. So the numerator is now what it was before. The denominators, rather than being raised to the power of m, are raised to the power of m minus 1. So clearly, this is the same thing as fm minus 1. So basically these functions have this nice character that they are like a ladder. You take a derivative of one and you get the one that is indexed by one less. OK?

AUDIENCE: Is there a factor of eta that comes out to be [INAUDIBLE].

PROFESSOR: In the derivatives?

AUDIENCE: I mean, it seems to me that there's an eta-- like, in that relation that--

PROFESSOR: In this relation?

AUDIENCE: Yeah.

PROFESSOR: OK. So let's do it by step. If I take a derivative, divide dz, I get 1 plus eta z squared z 2 to the m minus 1 plus z squared' 3 to the m minus 1. Then I multiply by z, this becomes z, this becomes z squared, this becomes z cubed. And you can see that the etas are exactly where they should be.

AUDIENCE: Would you explain why the index of summation starts from--

PROFESSOR: One?

AUDIENCE: --one.

PROFESSOR: OK. So let's take a look at this function. You can see there is a z inverse in the denominator. So I made it into a z here. So you can see that if z is very small, if will start with z. All right? Now what happened here? I wrote this as a sum of geometric series. I pulled out this factor of z that was in the numerator, and the denominator I wrote as a series that starts with 0.

But then there is this z out front. I pulled this z e to the minus x in front here and then it starts with 1. I could have written it-- I can put this in here and write it as alpha plus 1 with alpha starting from 0, et cetera. Or I can write it [INAUDIBLE]. Or, alternatively, you can say, let's put the eta here and eta here. And then I see that this is the geometric series where the first term is this z to the minus x [INAUDIBLE]. OK?

So I want to erase this board but keep this formula. So let me erase it and then rewrite it. So this I will rewrite as N lambda q over g, which is my degeneracy factor, is f 3/2 eta of z. And in the limit where z goes to small, I can use the expansion that I

just derived, which is z plus eta z squared 2 to the-- what is it-- 3/2 plus z cubed. 3 to the 3/2 and so forth.

You can see that precisely because as z goes to 0, this series goes to 0, that smallness of z is the same thing or implied by the smallness of z. But that z is equal to d only at the lowest order. So what happens at higher orders, z-- inverting this-- is d minus eta z squared 2 to the 3/2 plus z cubed 3 to the 3/2, and so forth.

And we saw this when I was doing the calculation for the interacting gas, that clearly to lowest order z equals to d, if I want to get the next order, I substitute the result of the previous order into this expression. So z I substitute for d, I will d minus eta d squared 2 to the 3/2 as the first contribution. And then something that is order of d cubed.

And if I want the answer to the d cubed order, I take this whole thing that is that order of d squared and substitute in this series. And so what do I get? I will get d minus eta 2 to the 3/2, square of z-- which is the square of this quantity-- square of this quantity is d squared minus twice the product of these two terms.

So I have twice eta d cubed 2 to the 3/2. And then square of this term, but that's order of e to the 4. I don't care for that. And then I have, from here, plus d cubed 3 to the 3/2 ignoring terms of the order of d to the 4th. OK? So z is--

AUDIENCE: [INAUDIBLE] z equals d minus and the d [INAUDIBLE].

PROFESSOR: Ah, great. Thank you. Minus. Minus. Good. Saves me having to go back later when the answer doesn't match. OK. Fine.

Now let's organize terms. The lowest order I have d. At order of d squared I have still just this one term, minus eta d squared over 2 to the 3/2. We knew that was the correct result at order of d squared.

At order of d cubed I have two terms. One is the product of these two terms. The etas disappear, minus 2, and this will give me 2 over 9 which is 1/4. And from other part, I have 1 over 3 to the 3/1 d cubed. And I haven't calculate things at order of d

to the 4th. OK?

Fine. So we have calculated in a power series that we can go to whatever order you like. This is z which is related to chemical potential as a function of this degeneracy that is simply related to density. Now, the formula for the pressure is up there. It is that eta P is g over lambda cubed f 5/2 eta of z. Right?

So reorganizing this a little bit, the dimensionless form is beta P lambda cubed over g f 5/2 eta of z. f 5/2 we can make an expansion. It is z plus eta z squared 2 to the 5/2 plus z cubed 3 to the 5/2 and so forth. For z I substitute from the line above. So I have d minus eta d squared 2 to the 3/2 plus 1/4 minus 1 over 3 to the 3/2 d cubed. So that's the first term.

The next term is eta 2 to the 5/2 z squared. So I have to square what is up here and here keeping terms to order of d cubed. So what do I have? I have d squared from squaring this term. And a term that is twice the product of these two terms. That's the only term that is order of d cubed. So I have minus 2 eta 2 to the 3/2 d cubed.

And finally, z cubed over 3 to the 5/2 would be d cubed over 3 to the 5/2. And I haven't calculated anything at all of d to the 4th. OK? So what do I have? I have that beta P lambda cubed over g. Well, you can see that everything is going to be proportional to d d squared d cubed. So let's put out the factor of d which is m lambda cubed over g. So this is just one factor of d that I have pulled out. So that then I start with 1, and then the next correction I have an eta d squared 2 to the 3/2 here, eta d squared 2 to the 5/2 here with opposite sign. So this is twice bigger than that.

So the sum total of them would be minus eta divided by 2 to the 5/2 times d squared-- one factor of d I have out front-- the other factor I will write here and lambda cubed over g. And then at order of d cubed, I have a bunch of terms. I have a 1/4 here, and from the product of these terms, I will have etas disappear, I have 2 divided by 2 to the 8/2 which is 2 to the 4th which is 16.

So the product of these things will give me a minus 1/8. I have 1/4 minus 1/8 I will

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get 1/8.

AUDIENCE: [INAUDIBLE].

PROFESSOR: OK. And then we have minus 3-- 1 over 3 to the 1/2 and this 3 to the 5/2. This is three times larger than this. This is 3 over 3 to the 5/2. Subtract one of them, I will be left minus 2 divided by 3 to the 5/2 [? N ?] lambda cubed over g squared. And so forth.

So you can see that the form that we have is beta P-- the pressure-- starts with density. I essentially divide it through by lambda q over g. I start with the ideal gas result, n, and then I have a correction that is order of n squared. I can write it as B2 n squared. In view up our [INAUDIBLE] co-efficients.

And we see that our B2 is minus eta times-- sorry. Minus eta 2 to the 5/2 times n lambda cubed over g. Remember that B2s, et cetera, had dimensions of volume. So if the volume is provided by lambda cubed, it is an additional factor. For fermions it's a positive pressure. So the pressure goes up because fermions repel each other. For bosons it is negative. Bosons attract each other.

And we have actually calculated this. We did a calculation of the canonical form where we had the one exchange corrections. And if you go back to the notes or if you remember, we had calculated precisely this. But then I said that if I went and tried to calculate the next correction using the canonical formulation, I would have to keep track of diagrams that involve three exchanges and all kinds of things and it was complicated.

But in this way of looking at things, you can see that we can very easily compute the [INAUDIBLE] co-efficients and higher orders. In particular, the third [INAUDIBLE] co-efficient we compute to be 1/8 minus 2 3 to the 5/2 lambda cubed over g squared. And you can keep going and calculate higher and higher order corrections.

AUDIENCE: Question.

PROFESSOR: Yes.

AUDIENCE: So if you're doing this expansion for the z being close to 0--

PROFESSOR: Yes.

AUDIENCE: --it appears that chemical potential is a large negative number.

PROFESSOR: Yes.

AUDIENCE: What kind of interaction does it imply between our grand canonical and [INAUDIBLE] and the system from which it takes particles? Because chemical potential represents energy threshold for the particle to move from there to--

PROFESSOR: OK. So as you say, it corresponds to being large and negative. Which means that, essentially, I think the more kind of physical way of looking at it is-- this z is e to the beta mu-- is the likelihood of grabbing and adding one particle to your system. And what we have is that this is much, much less than 1. And the way to achieve that is to say that it is very hard to extract and bring in other particle from whatever reservoir you have. And because it is so hard that you have such low density of particles in the system that you have.

So for example, if you are in vacuum and you put a surface, and some atoms from the surface detach and form a gas in your container, the energetics of binding those atoms to the substrate particles was huge and few of them managed to escape. OK?

All right. But this is only one limit. And this is the limit that we already knew. We are really interested in the other limit where we are quantum dominated. And clearly we want to go to the limit where the argument here is not small. I have to include a large number of terms in this series. And then I have to figure out what happens. I mean, how do I have to treat the cases where I want to solve this equation and the left hand side, d, is a large number?

So how can I make the value of the function large? What does that imply about argument of the function? OK? So different functions will become large for different values of the argument. And as we will show for the case of bosons, these functions

become large where the argument becomes 1. Whereas, for fermions, they become large when the argument itself becomes very large.

So the behavior of these functions is very different. When they become large they become large at different points when we are thinking about bosons and fermions. And because of that, at this stage I have to separately address the two problems. So up to here, the high temperature limit, we sort of started from the same point and branched off slightly by going into boson and fermion direction. But when I go to the other limit, I have to discuss the two cases separately. So let's start with the case of fermions.

So degenerate-- degenerating a sense of d being larger than 1 rather than character flaws. So what do we do in this limit? So what I need to know is to look at one of these functions. So let's say d n lambda cubed over g is f 3/2 minus of z which is 1 over this 1/2 factorial integral 0 to infinity dx x to the 1/2. And here I have a plus because I'm looking at the fermionic version of these. I have z inverse e to the x and I can write that as e to the x minus log z. OK?

Now, essentially, if I look at this function-- actually, let's look at the function that is just 1 over e to the x minus log z plus 1 as a function of x. What does it do in the limit where z is large? In the limit where z is large, log z is also very large. And then what I have is something that if x is much less than log z-- essentially, this factor will be 0 and I have 1. Whereas if x is much greater than log z, then this is exponentially large quantity and I will have 0.

So basically, at around log z, this switches from being very close to 0 to being 1. Indeed, that x equals 2 log z is exactly 1/2. So here is 1/2. It is switching very rapidly. And you can see that, essentially, the value of the function depends on the combination x minus log z. So if I were to make log z much larger, I'm essentially transporting this combination without changing its shape, either in one direction or the other direction. OK?

And if I go to the limit, essentially that log z is very large, I will have 1 for a huge distance and then, at some point, I would switch to 0. And what I'm doing, therefore,

is to essentially approximately integrating from 0 to log z x to the 1/2 dx. Which would give me x to the 3/2-- sorry, log z to the 3/2 over 3/2.

And you can see that I can repeat this for any other value of m here. I would be integrating x to the m minus 1. So you would conclude that the limit as z is much, much larger than 1, of one of these functions fm minus of z, is obtained by integrating 0 to log z dx x to the m. I will get another factor of log z. And then I will get log z to the power of m divided by 1 over m multiplying by 1 over m minus 1 factorial. It will give me m factorial.

Now, of course there will be correction to this because this function would be equal to this if I was integrating a step function. And I don't have a step function. I have something that gradually vanishes. Although, the interval over which it vanishes is scaling much smaller than log z.

So what you can do, is you can do some manipulations and calculate corrections to this formula that become significant as log z is not exactly infinite but becomes smaller and smaller. It turns out that that will generate for you a series that is inverse powers of log z. And in fact, inverse even powers of log z. So that the next term will fall off as 1 over log z squared. The next term will fall off as 1 over z to the fourth and so forth.

And the coefficient of the first one-- and I show you this in the notes, how to calculate it-- is pi squared over 6 times m m minus 1. The coefficient of the next term is 7 pi to the fourth 360 m m minus 1, m minus 2, m minus 3. And there's a formula for subsequent terms in the series, how they are generated. And this thing is known as the Sommerfeld expansion. OK?

So, essentially, if you plot as a function of z one of these fm minuses of z's, what we have according to the previous expansion, as you go to z equals to 0 you have a linear function z. And then the next correction is minus z squared. You start to bend in this direction. And then you have higher order corrections.

What we see, that at larger values, it keeps growing. And the growth at larger

values is this. So here you have z minus z squared 2 to the power of m. Out here you have log z to the m over m factorial. And then this corrections from the Sommerfeld expansion. OK?

And so once we have that function, which is the right hand side of this equality, for a particular density we are asking, what is the value of z? And at low densities we are evaluating solutions here. At high values of density we are picking up solutions on this other. OK?

So let's imagine that we are indeed in that limit. So what do we have? We have that n lambda cubed over g is log z to the 3/2 over 3/2 factorial. That's the leading term. Sub-leading term is 1 plus pi squared over 6. m, in our case, is 3/2. m minus 1 would be 1/2 divided by log z squared. I will stop at this order. That's all we will need.

And so I will use this to calculate z as a function n. Again, same procedure as before. I can invert this and write this as log z being 3/2 factorial n lambda cubed over g. so I multiplied by this. I have to raise to the 2/3 power-- the inverse of 3/2. I take this combination to the other side. So I have 1. This becomes plus pi squared over 8 1 over log z squared raised to the minus 2/3 power because I had to take it to the other side. OK.

So at the lowest order, what I need to do is to calculate log z from the zeroed order term and then, at the next order, substitute that value of log z in here and get the next correction. So the lowest order correction will tell me that log z--- which, remind you, is simply beta mu-- is this combination raised to the 2/3 power.

What is that communication? 3/2 factorial. 3/2 factorial we saw was 3/2 1/2 factorial-- which was root pi over 2-- so this is 3/2 factorial. I have n over g. Oh, sorry. n lambda cubed. So write it in this fashion. n over g. And then I have lambda cubed. Let me put this to the 2/3 power because lambda cubed to the 2/3 power is simply lambda squared. And lambda squared is going to be h squared divided by 2 pi mkT. So this is lambda cubed.

What I'm going to do is notice that one of these h over 2pi's is in fact already an h bar. I'd like to write the result in terms of h bar squared divided by 2m. The 1 over kT I can write as beta. Ultimately, it is my intention to cancel that beta which is the 1 over kT in the beta that I have out here. And I get the value for mu. OK?

To go from h squared over 2pi to h bar squared over 2m, I need a factor of 4pi squared. Those 4pi squared I will put out front in here. And they will change the result to something that is like this. It is n over g is going to be remaining there. I have pi by the time it gets in front over here becomes pi to the 3/2. Pi to the 3/2 multiplied by pi to the 1/2 becomes pi squared.

You can convince yourself that the factors of 2 and all of those things manage to give you something that is 6pi squared n over g to the 2/3 power. Yes?

AUDIENCE: So when [INAUDIBLE] chemical potential as no temperature [INAUDIBLE].

PROFESSOR: Exactly. When you go to 0 temperature, the chemical potential of a Fermi gas goes to a constant that you have actually seen-- and I just was about to make it clear to you what that constant is. So we get that limit as essentially this T goes to 0, which is another way of achieving the combination n lambda cubed over g goes to infinity. When T goes to 0, lambda goes to infinity, this combination becomes large. Canceling the betas, the limit of mu is something that potentially in solid state courses you have seen as the fermion energy.

And the fermion energy is h bar squared over 2m. Some Fermi momentum squared. And hopefully the Fermi momentum is going to come out to be 6pi squared n over g to the 1/3. And one way that you usually get your Fermi momentum is you say that, what I need to do is to fill up all of this state up to some value of kf. And the number of states that there are is the volume of the sphere- 4pi over 3k f cubed, of course times the volume times the degeneracy coming from spins if I have multiple possibilities. And this should give me the total number n.

Ah. I made some big mistake here. Can somebody point it out to me? How do we calculate Fermi momentum and Fermi [INAUDIBLE]? Not coming out the answer I

want.

AUDIENCE: One [INAUDIBLE] is a sphere, right?

PROFESSOR: No. It's a sphere.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Thank you. Divide by density of state. I multiplied by the V but I forgot the 2pi cubed here. OK? So when you divide 4pi divided by 8pi cubed, you will have 2pi squared times 3, which will give you 6pi squared. n over V is the density. Divided by g. So you can see that we get the right result. Yes?

AUDIENCE: [INAUDIBLE] cubed.

PROFESSOR: f cubed. Yes. That's right. So kf is 1/3 of that. Let's go step by step. A is 6pi squared n over g to the 1/3. kf squared would be the 2/3 power [? of g. ?] Yes?

AUDIENCE: So you said that this is the lowest order term for the chemical potential.

PROFESSOR: Exactly. Yes.

AUDIENCE: But aren't there going to be maybe a second order temperature dependent term?

PROFESSOR: Yes. So when we substitute that formula in here, we will find the value of the chemical potential. So I will draw it for you and then recalculate the result later. So this is the chemical potential of the Fermi system as a function of its temperature.

What did we establish so far? We established that at zero temperature it is in fact epsilon f. We had established that at high temperature-- this was the discussion we had before-- it's large and negative. It is down here. So indeed the behavior of the chemical potential of the Fermi gas as a function of temperature is something like this. And so the chemical potential at zero temperature is epsilon f. As you go to finite temperature, it will get reduced by a factor that I will just calculate. OK?

So if I now substitute this result in here to get the next correction, what do I have? OK. So this zero to order term we established is the same thing as beta epsilon f.

And then the next order term will be 1 plus-- this log z is going to be small-- minus 2/3 times this is going to give me minus pi squared over 6. Right?

And then I have 1 over log z squared. One over log z is simply 1 over beta epsilon f which is the same thing as kT over epsilon f squared So what we see is that the chemical potential starts to be epsilon f and then it starts to go down parabolically in temperature. And the first correction is of the order of kT over epsilon f.

Not surprisingly, it will go to 0 at the value T that is off the order of epsilon f over kT. So the chemical potential of the Fermi gas actually changes sign. It's negative and large at high temperature, it goes to the Fermi energy which is positive at low temperatures.

So once we know this behavior or the chemical potential, we can calculate what the energy and pressure on. And again, at zero temperature you will have some energy which comes from filling out this Fermi C. And then we will get corrections to energy and pressure that we can calculate using the types of formulas that we have that relate pressure and energy to f 5/2 of z. And so we can start to calculate the corrections that you have to completely Fermi sphere. And how those corrections give you, say, the linear heat capacity is something that we will derive next time.