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PROFESSOR: So the question is, how do we describe the various motions that are taking place? And in principle, the Boltzmann equation that we have been developing should be able to tell us something about all of that. Because we put essentially all of the phenomena that we think are relevant to a dilute gas, such as what's going on in this room, into that equation, and reduced the equation to something that had a simple form where on the left hand side there was a set of derivatives acting on the one particle, probability as a function of position, and momentum, let's say, for the gas in this room.

And on the right hand side, we had the collision operator. And again, to get the notation straight, this set of operations on the left hand side includes a time derivative, a thing that involves the velocity causing variations in the coordinate, and any external forces causing variations in momentum.

I write that because I will, in order to save some space, use the following notation in the remainder of this lecture. I will use del sub t to indicate partial derivatives with respect to time. This term I will indicate as p alpha over m d alpha. So d sub alpha stands for derivative with respect to, say, x, y, and z-coordinates of position. And this summation over the repeated index is implied here.

And similarly here, we would write F alpha d by dp alpha. We won't simplify that. So basically, L is also this with the summation implicit.

Now, the other entity that we have, which is the right hand side of the equation, had something to do with collisions. And the collision operator Cff was an integral that involved bringing in a second coordinate, or second particle, with momentum p2, would come from a location such that the impact parameter would be indicated by

1

B. We would need to know the flux of incoming particles. So we had the relative velocities.

And then there was a term that was essentially throwing you off the channel that you were looking at. And let's say we indicate the variable here rather than say p by p1. Then I would have here F evaluated at p1, F evaluated at p2, p2 being this momentum that I'm integrating. And then there was the addition from channels that would bring in probability.

So p1 prime and p2 prime, their collision would create particles in the channel p1, p2. And there was some complicated relation between these functions and these functions for which you have to solve Newton's equation. But fortunately, for our purposes, we don't really need all of that. Again, all of this is evaluated at the same location as the one that we have over here.

Now let's do this. Let's take this, which is a function of some particular q and some particular p1, which is the things that we have specified over there. Let's multiply it with some function that depends on p1, q, and potentially t, and integrate it over P1.

Once I have done that, then the only thing that I have left depends on q and t, because everything else I integrated over. But in principle, I made a different integration. I didn't have the integration over q. So eventually, this thing will become a function of q.

OK, so let's do the same thing on the right hand side of this equation. So what I did was I added the integration over p1. And I multiplied by some function. And I'll remember that it does depend on q. But since I haven't written the q argument, I won't write it here. So it's chi of p1.

So I want to-- this quantity j that I wrote is equal to this integral on the right. Now, we encountered almost the same integral when we were looking for the proof of the H-theorem where the analog of this chi of p1 was in fact log of f evaluated at p1. And we did some manipulations that we can do over here.

First of all, here, we have dummy integration variables p1 and p2. We can just

change their name and then essentially average over those two possibilities. And the other thing that we did was this equation has the set of things that come into the collision, and set the things that, in some sense, got out of the collision, or basically things that, as a result of these collisions, create these two.

So you have this symmetry between the initiators and products of the collision. Because essentially the same function describes going one way and inverting things and going backwards. And we said that in principle, I could change variables of integration. And the effect of doing that is kind of moving the prime coordinates to the things that don't have primes.

I don't know how last time I made the mistake of the sign. But it's clear that if I just put the primes from here to here, there will be a minus sign. So the result of doing that symmetrization should be a minus chi of p1 prime minus chi of p2 prime. And again, to do the averaging, I have to put something else.

Now, this statement is quite generally true. Whatever chi I choose, I will have this value of j as a result of that integration. But now we are going to look at something specific. Let's assume that we have a quantity that is conserved in collision. This will be 0 for collision conserved quantity.

Like let's say if my chi that I had chosen here was some component of momentum px, then whatever some of the incoming momenta will be the sum of the outgoing momenta. So essentially anything that I can think of that is conserved in the collision, this function that relates p primes to p1 and p2 has the right property to ensure that this whole thing will be 0.

And that's actually really the ultimate reason. I don't really need to know about all of these cross sections and all of the collision properties, et cetera. Because my focus will be on things that are conserved in collisions. Because those are the variables that are very slowly relaxing, and the things that I'm interested in. So what you have is that for these collision conserved quantities, which is the things that I'm interested in, this equation is 0.

Now, if f satisfies that equation, I can certainly substitute over here for Cff Lf. So if f satisfies that equation, and I pick a collision conserved quantity, the integral over p1 of that function of the collision conserved quantity times the bunch of first derivatives acting on f has to be 0.

So this I can write in the following way-- 0 is the integral over p1. Actually, I have only one momentum. So let's just ignore it from henceforth. The other momentum I just introduced in order to be able to show that when integrated against the collision operator, it will give me 0.

I have the chi, and then this bunch of derivatives dt plus p alpha over m p alpha plus f alpha d by dp alpha acting on f. And that has to be 0.

STUDENT: So alpha stands for x, y, z?

PROFESSOR: Yes, alpha stands for the three components x, y, and z throughout this lecture. And summation over a repeated index is assumed. All right, so now what I want to do is to move this chi so that the derivatives act on both of them. So I'll simply write the integral of dqp-- if you like, this bunch of derivatives that we call L acting on the combination chi f. But a derivative of f chi gives me f prime chi. But also it keeps me chi prime f that I don't have here. So I have to subtract that.

And why did I do that? Because now I end up with integrals that involve integrating over f against something. So let's think about these typical integrals. If I take the integral over momentum of f of p and q and t-- remember, f was the one particle density. So I'm integrating, let's say, at a particular position in space over all momentum.

So it says, I don't care what momentum. I just want to know whether there's a particle here. So what is that quantity? That quantity is simply the density at that location.

Now suppose I were to integrate this against some other function, which could depend on p, q, and t, for example? I use that to define an average. So this is going to be defined to be the average of O at that location q and t. So for example, rather

than just calculating whether or not there are particles here, I could be asking, what is the average kinetic energy of the particles that are here? Then I would integrate this against p squared over 2m. And this average would give me the local expectation value of p squared over m, just a normalization n so that it's appropriately normalized.

So with this definition, I can write the various terms that I have over here. So let me write it a little bit more explicitly. What do we have? We have integral d cubed p. We have this bunch of derivatives acting on chi f minus f times this bunch of derivatives acting on chi.

So let's now look at things term by term. The first term is a time derivative. The time derivative I can take outside the integral. Once I take the time derivative outside the integral, what is left? What is left is the integral of chi f, exactly what I have here. O has been replaced by chi. So what I have is the time derivative of n expectation value of chi using this definition.

Let's look at the next term. The next term, these derivatives are all over position. The integration is over momentum. I can take it outside. So I can write it as d alpha. And then I have a quantity that I'm integrating against f. So I will get n times this local average of that quantity. What's that quantity? It's p alpha over m times chi.

What's the third term? The third term is actually an integral over momentum. But I'm integrating over momentum. So again, you can sort of remove things to boundaries and convince yourself that that integral will not give you a contribution.

The next bunch of terms are simply directly this-- f integrated against something. So they're going to give me minus n times the various averages involved-- d t chi minus n, the average of p alpha over m, the alpha chi. And then f alpha I can actually take outside, minus n f alpha, the average of d chi by d p alpha. And what we've established is that that whole thing is 0 for quantities that are conserved under collisions.

So why did I do all of that? It's because solving the Boltzmann equation in six

dimensional phase space with all of its integrations and derivatives is very complicated. But really, the things that are slowly relaxing are quantities that are conserved collisions, such as densities, average momentum, et cetera. And so I can focus on variations of these through this kind of equation.

Essentially, what that will allow me to do is to construct what are known as hydrodynamic equations, which describe the time evolution of slow variables of your system, the variables that are kind of relevant to making thermodynamic observations, as opposed to variables that you would be interested in if you're thinking about atomic collisions.

So what I need to do is to go into that equation and pick out my conserved quantities. So what are the conserved quantities, and how can I describe them by some chi? Well, we already saw this when we were earlier on trying to find some kind of a solution to the H by dt equals 0. We said that log f has to be the sum of collision conserved quantities. And we identified three types of quantities.

One of them was related to the number conservation. And essentially, what you have is that 1 plus 1 equals to 1 plus 1. So it's obvious. The other is momentum. And there are three components of this-- px, py, pz.

And the third one is the kinetic energy, which is conserved in collisions. In a potential, clearly the kinetic energy of a particle changes as a function of position. But within the short distances of the collisions that we are interested in, the kinetic energy is a conserved quantity.

So my task is to insert these values of chi into that equation and see what information they tell me about the time evolution of the corresponding conserved quantities. So let's do this one by one. Let's start with chi equals to 1.

If I put chi equals to 1, all of these terms that involve derivatives clearly needed to vanish. And here, I would get the time derivative of the density. And from here, I would get the alpha n expectation value of p alpha over m. We'll give that a name. We'll call that u alpha. So I have introduced u alpha to be the expectation value of p

alpha over m. And it can in principle depend on which location in space you are looking at.

Somebody opens the door, there will be a current that is established. And so there will be a local velocity of the air that would be different from other places in the room. And that's all we have. And this is equal to 0. And this is of course the equation of the continuity of the number of particles. You don't create or destroy particles. And so this density has to satisfy this nice, simple equation.

We will sometimes rewrite this slightly in the following way. This is the derivative of two objects. I can expand that and write it as dt of n plus, let's say, u alpha d alpha of n. And then I would have a term that is n d alpha u alpha, which I will take to the other side of the equation.

Why have I done that? Because if I think of n as a function of position and time-- and as usual, we did before define a derivative that moves along this streamline-- you will have both the implicit time derivative and the time derivative because the stream changes position by an amount that is related to velocity.

Now, for the Liouville equation, we have something like this, except that the Liouville equation, the right hand side was 0. Because the flows of the Liouville equation, the Hamiltonian flows, were divergenceless. But in general, for a compressable system, such as the gas in this room, the compressibility is indicated to a nonzero divergence of u. And there's a corresponding term on the right hand side.

So that's the first thing we can do. What's the second thing we can do? I can pick p-- let's say p beta, what I wrote over there. But I can actually scale it. If p is a conserved quantity, p/m is also a conserved quantity.

Actually, as far as this chi is concerned, I can add anything that depends on q and not p. So I can subtract the average value of this quantity. And this is conserved during collisions. Because this part is the same thing as something that is related to density. It's like 1 plus 1 equals to 1 plus 1. And p over beta is conserved also.

So we'll call this quantity that we will use for our candidate chi as c beta. So

essentially, it's the additional fluctuating speed, velocity that the particles have, on top of the average velocity that is due to the flow. And the reason maybe it's useful to do this is because clearly the average of c is 0. Because the average of p beta over m is u beta. And if I do that, then clearly at least the first thing in the equation I don't have to worry about. I have removed one term in the equation.

So let's put c beta for chi over here. We said that the first term is 0. So we go and start with the second term. What do I have? I have d alpha expectation value of c beta.

And then I have p alpha over m. Well, p alpha over m is going to be u alpha plus c alpha. So let's write it in this fashion-- u alpha plus c alpha for p alpha over m. And that's the average I have to take.

Now let's look at all of these terms that involve derivatives. Well, if I want to take a time derivative of this quantity, now that I have introduced this average, there is a time derivative here. So the average of the time derivative of chi will give me the time derivative from u beta.

And actually the minus sign will cancel. And so I will have plus n, the expectation value of d-- well, there's no expectation value of something like this. When I integrate over p, there's no p dependence. So it's just itself. So it is n dt of u beta.

OK, what do we have for the next term? Let's write it explicitly. I have n. p alpha over m I'm writing as u alpha plus c alpha. And then I have the position derivative of c beta. And that goes over here. So I will get d alpha of u beta with a minus sign. So this becomes a plus.

The last term is minus n f alpha. And I have to take a derivative of this object with respect to p alpha. Well, I have a p here. The derivative of p beta with respect to p alpha gives me delta alpha beta. So this is going to give me delta alpha beta over m. And the whole thing is 0.

So let's rearrange the terms over here. The only thing that I have in the denominator is a 1/m. So let me multiply the whole equation by m and see what

happens. This term let's deal with last. This term, the first term, becomes nmdt of u beta. The change in velocity kind of looks like an acceleration.

But you have to be careful. Because you can only talk about acceleration acting for a particle. And the particle is moving with the stream. OK, and this term will give me the appropriate derivative to make it a stream velocity.

Now, when I look at this, the average of c that appears here will be 0. So the term that I have over there is u alpha d alpha u beta. There's no average involved. It will give me n. m is common, so I will get u alpha d alpha u beta, which is nice. Because then I can certainly regard this as one of these stream derivatives.

So these two terms, the stream derivative of velocity with time, times mass, mass times the density to make it mass per unit volume, looks like an acceleration. So it's like mass times acceleration. Newton's law, it should be equal to the force.

And what do we have if we take this to the other side of the equation? We have f beta. OK, good, so we have reproduced Newton's equation. In this context, if we're moving along with the stream, mass times the acceleration of the group of particles moving with the stream is the force that is felt from the external potential.

But there's one other term here. In this term, the term that is uc will average to 0, because the average of u is 0. So what I will have is minus-- and I forgot to write down an n somewhere here. There will be an n. Because all averages had this additional n that I forgot to put.

I will take it to the right hand side of the equation. And it becomes d by dq alpha of n. I multiply the entire equation by m. And then I have the average of c alpha c beta. So what happened here? Isn't force just the mass times acceleration? Well, as long as you include all forces involved.

So if you imagine that this room is totally stationary air, and I heat one corner here, then the particles here will start to move more rapidly. There will be more pressure here. Because pressure is proportional to temperature, if you like. There will be

more pressure, less pressure here. The difference in pressure will drive the flow. There will be an additional force.

And that's what it says. If there's variation in these speeds of the particles, the change in pressure will give you a force. And so this thing, p alpha beta, is called the pressure tensor. Yes.

STUDENT: Shouldn't f beta be multiplied by n, or is there an n on the other side of that?

PROFESSOR: There is an n here that I forgot, yes. So the n was in the first equation, somehow got lost.

STUDENT: So the pressure is coming from the local fluctuation?

PROFESSOR: Yes. And if you think about it, the temperature is also the local fluctuation. So it has something to do with temperature differences. Pressure is related to temperature. So all the things are connected. And in about two minutes, I'll actually evaluate that for you, and you'll see how. Yes.

STUDENT: Is the pressure tensor distinct from the stress tensor?

PROFESSOR: It's the stress tensor that you would have for a fluid. For something more complicated, like an elastic material, it would be much more complicated-- not much more complicated, but more complicated. Essentially, there's always some kind of a force per unit volume depending on what kind of medium you have. And for the gas, this is what it is. Yes.

STUDENT: So a basic question. When we say u alpha is averaged, averaged over what? Is it by the area?

PROFESSOR: OK, this is the definition. So whenever I use this notation with these angles, it means that I integrated over p. Why do I do that? Because of this asymmetry between momenta and collision and coordinates that is inherent to the Boltzmann equation.

When we wrote down the Liouville equation, p and q were completely equivalent. But by the time we made our approximations and we talked about collisions, et

cetera, we saw that momenta quickly relax. And so we can look at the particular position and integrate over momenta and define averages in the sense of when you think about what's happening in this room, you think about the wind velocity here over there, but not fluctuations in the momentum so much, OK?

All right, so this clearly is kind of a Navier-Stokes like equation, if you like, for this gas. That tells you how the velocity of this fluid changes. And finally, we would need to construct an equation that is relevant to the kinetic energy, which is something like p squared over 2m.

And we can follow what we did over here and subtract the average. And so essentially, this is kinetic energy on top of the kinetic energy of the entire stream. This is clearly the same thing as mc squared over 2, c being the quantity that we defined before.

And the average of mc squared over 2 I will indicate by epsilon. It's the heat content. Or actually, let's say energy density. It's probably better. So now I have to put mc squared over 2 in this equation for chi and do various manipulations along the lines of things that I did before. I will just write down the final answer.

So the final answer will be that dt of epsilon. We've defined dt. I move with the streamline. So I have dt plus u alpha d alpha acting on this density, which is a function of position and time. And the right hand side of this will have two terms.

One term is essentially how this pressure kind of moves against the velocity, or the velocity and pressure are kind of hitting against each other. So it's kind of like if you were to rub two things-- "rub" was the word I was looking. If you were to rub two things against each other, there's heat that is generated.

And so that's the term that we are looking at. So what is this u alpha beta? u alpha beta-- it's just because p alpha beta is a symmetric object. It doesn't make any difference if you exchange alpha and beta. You symmetrize the derivative of the velocity. And sometimes it's called the rate of strain.

And there's another term, which is minus 1/n d alpha of h alpha. And for that, I need

to define yet another quantity, this h alpha, which is nm over 2 the average of 3 c's, c squared and then c alpha. And this is called the heat transport.

So for a simpler fluid where these are the only conserved quantities that I have, in order to figure out how the fluid evolves over time, I have one equation that tells me about how the density changes. And it's related to the continuity of matter. I have one equation that tells me how the velocity changes.

And it's kind of an appropriately generalized version of Newton's law in which mass times acceleration is equated with appropriate forces. And mostly we are interested in the forces that are internally generated, because of the variations in pressure. And finally, there is variations in pressure related to variations in temperature. And they're governed by another equation that tells us how the local energy density, local content of energy, changes as a function of time. So rather than solving the Boltzmann equation, I say, OK, all I need to do is to solve these hydrodynamic equations. Question?

STUDENT: Last time for the Boltzmann equation, [INAUDIBLE].

PROFESSOR: What it says is that conservation laws are much more general. So this equation you could have written for a liquid, you could have written for anything. This equation kind of looks like you would have been able to write it for everything. And it is true, except that you wouldn't know what the pressure is. This equation you would have written on the basis of energy conservation, except that you wouldn't know what the heat transport vector is.

So what we gained through this Boltzmann prescription, on top of what you may just guess on the basis of conservation laws, are expressions for quantities that you would need in order to sort these equations, because of the internal pressures that are generated because of the way that the heat is flowing.

STUDENT: And this quantity is correct in the limit of--

PROFESSOR: In the limit, yes. But that also really is the Achilles' heel of the presentation I have given to you right now. Because in order to solve these equations, I should be able

to put an expression here for the pressure, and an expression here for the h. But what is my prescription for getting the expression for pressure and h? I have to do an average that involves the f. And I don't have the f.

So have I gained anything, all right? So these equations are general. We have to figure out what to do for the p and h in order to be able to solve it. Yes.

STUDENT: In the last equation, isn't that n epsilon instead of epsilon?

PROFESSOR: mc squared over 2-- I guess if I put here mc squared over 2, probably it is nf.

STUDENT: I think maybe the last equation it's n epsilon, the equation there.

PROFESSOR: This equation is OK. OK, so what you're saying-- that if I directly put chi here to be this quantity, what I would need on the left hand side of the equations would involve derivatives of n epsilon. Now, those derivatives I can expand, write them, let's say, dt of n epsilon is epsilon dt of n plus ndt of epsilon.

And then you can use these equations to reduce some of that. And the reason that I didn't go through the steps that I would go from here to here is because it would have involved a number of those cancellations. And it would have taken me an additional 10, 15 minutes.

All right, so conceptually, that's the more important thing. We have to find some way of doing these things. Now, when I wrote this equation, we said that there is some kind of a separation of time scales involved in that the left hand side of the equation, the characteristic times are order of the time it takes for a particle to, say, go over the sides of the box, whereas the collision times, 1 over tau x, are such that the right hand side is much larger than the left hand side.

So as a 0-th order approximation, what I will assume is that the left hand side is so insignificant that I will set it to 0. And then my approximation for the collision is the thing that essentially sets this bracket to 0. This is the local equilibrium that we wrote down before.

So that means that I'm assuming a 0-th order approximation to the solution of the Boltzmann equation. And very shortly, we will improve up that. But let's see what this 0-th order approximation gives us, which is-- we saw what it is. It was essentially something like a Gaussian in momentum.

But the coefficient out front of it was kind of arbitrary. And now that I have defined the integral over momentum to be density, I will multiply a normalized Gaussian by the density locally. And I will have an exponential.

And average of p I will shift by an amount that depends on position. And I divide by some parameter we had called before beta. But that beta I can rewrite in this fashion. So I have just rewritten the beta that we had before that was a function of q and t as 1 over kBT. And this has to be properly normalized. So I will have 2 pi mkBT, which is a function of position to the 3/2.

And you can check that the form that I have written here respects the definitions that I gave, namely that if I were to integrate it over momentum, since the momentum part is a normalized Gaussian, I will just get the density. If I were to calculate the average of p/m, I have shifted the Gaussian appropriately so that the average of p/m is the quantity that I'm calling u.

The other one-- let's check. Essentially what is happening here, this quantity is the same thing as mc squared over 2kT if I use the definition of c that I have over there. So it's a Gaussian weight. And from the Gaussian weight, you can immediately see that the average of c alpha c beta, it's in fact diagonal. It's cx squared, cy squared. So the answer is going to be delta alpha beta. And for each particular component, I will get kT over m.

So this quantity that I was calling epsilon, which was the average of mc squared over 2, is essentially multiplying this by m/2 and summing over delta alpha alpha, which gives me a factor of 3. So this is going to give me 3/2 kT. So really, my energy density is none other than the local 3/2 kT. Yes?

STUDENT: So you've just defined, what is the temperature. So over all previous derivations, we

didn't really use the classical temperature. And now you define it as sort of average kinetic energy.

PROFESSOR: Yeah, I have introduced a quantity T here, which will indeed eventually be the temperature for the whole thing. But right now, it is something that is varying locally from position to position. But you can see that the typical kinetic energy at each location is of the order of kT at that location.

And the pressure tensor p alpha beta, which is nm expectation value of c alpha c beta, simply becomes kT over m-- sorry, nKT delta alpha beta. So now we can sort of start. Now probably it's a better time to think about this as temperature. Because we know about the ideal gas type of behavior where the pressure of the ideal gas is simply density times kT.

So the diagonal elements of this pressure tensor are the things that we usually think about as being the pressure of a gas, now at the appropriate temperature and density, and that there are no off diagonal components here. I said I also need to evaluate the h alpha. h alpha involves three factors of c.

And the way that we have written down is Gaussian. So it's symmetric. So all odd powers are going to be 0. There is no heat transport vector here.

So within this 0-th order, what do we have? We have that the total density variation, which is dt plus u alpha d alpha acting on density, is minus nd alpha u alpha. That does not involve any of these factors that I need.

This equation-- let's see. Let's divide by mn. So we have Dt of u beta. And let's again look at what's happening inside the room. Forget about boundary conditions at the side of the box. So I'm going to write this essentially for the case that is inside the box. I can forget about the external force. And all I'm interested in is the internal forces that are generated through pressure.

So this is dt plus u alpha d alpha of u beta. I said let's forget the external force. So what do we have? We have the contribution that comes from pressure. So we have minus the alpha. I divided through by nm. So let me write it correctly as 1 over nm. I

have the alpha. My pressure tensor is nkT delta alpha beta.

Delta alpha beta and this d alpha, I can get rid of that and write it simply as d beta. So that's the equation that governs the variations in the local stream velocity that you have in the gas in response to the changes in temperature and density that you have in the gas.

And finally, the equation for the energy density, I have dt plus u alpha d alpha. My energy density is simply related to this quantity T. So I can write it as variations of this temperature in position. And what do I have on the right hand side?

I certainly don't have the heat transport vectors. So all I have to do is to take this diagonal p alpha beta and contract it with this strain tensor u alpha beta. So the only term that I'm going to get after contracting delta alpha beta is going to be d alpha u alpha.

So let's make sure that we get the factors right. So I have minus p alpha beta is nkT d alpha u alpha. So now we have a closed set of equations. They tell me how the density, temperature, and velocity vary from one location to another location in the gas. They're completely closed. That's the only set of things that come together.

So I should be able to now figure out, if I make a disturbance in the gas in this room by walking around, by talking, by striking a match, how does that eventually, as a function of time, relax to something that is uniform? Because our expectation is that these equations ultimately will reach equilibrium. That's essentially the most important thing that we deduce from the Boltzmann equation, that it was allowing things to reach equilibrium. Yes.

STUDENT: For the second equation, that's alpha? The right side of the second equation?

PROFESSOR: The alpha index is summed over.

STUDENT: The right side. Is it the derivative of alpha or beta? Yeah, that one.

PROFESSOR: It is beta. Because, you see, the only index that I have left is beta. So if it's an index

by itself, it better be beta. How did this index d alpha become d beta? Because the alpha beta was delta alpha beta.

STUDENT: Also, is it alpha or is it beta?

PROFESSOR: When I sum over alpha of d alpha delta alpha beta, I get d beta. Yes.

STUDENT: Can I ask again, how did you come up with the f0? Why do you say that option?

PROFESSOR: OK, so this goes back to what we did last time around. Because we saw that when we were writing the equation for the hydt, we came up with a factor of what that was-- this multiplying the difference of the logs. And we said that what I can do in order to make sure that this equation is 0 is to say that log is additive in conserved quantities, so log additive in conserved quantities.

I then exponentiate it. So this is log of a number. And these are all things that are, when I take the log, proportional to p squared and p, which are the conserved quantities.

So I know that this form sets the right hand side of the Boltzmann equation to 0. And that's the largest part of the Boltzmann equation. Now what happens is that within this equation, some quantities do not relax to equilibrium. Some-- let's call them variations. Sometimes I will use the word "modes"-- do not relax to equilibrium.

And let's start with the following. When you have a sheer velocity-- what do I mean by that? So let's imagine that you have a wall that extends in the x direction. And along the y direction, you encounter a velocity field. The velocity field is always pointing along this direction. So it only has the x component. There's no y component or z component. But this x component maybe varies as a function of position.

So my ux is a function of y. This corresponds to some kind of a sheer. Now, if I do that, then you can see that the only derivatives that would be nonzero are derivatives that are along the y direction. But this derivative along the y direction in all of these equations has to be contracted typically with something else. It has to be

contacted with u.

But the u's have no component along the y direction. So essentially, all my u's would be of this form. Basically, there will be something like uy. Something like this would have to be 0.

You can see that if I start with an initial condition such as that, then the equations are that dt of n-- this term I have to forget-- is 0. Because for this, I need a divergence. And this flow has no divergence. And similarly over here what I see as dt of the temperature is 0. Temperature doesn't change.

And if I assume that I am under circumstances in which the pressure is uniform, there's also nothing that I would get from here. So essentially, this flow will exist forever. Yes.

STUDENT: Why does your u alpha d alpha n term go away? Wouldn't you get a uxdxn?

PROFESSOR: OK, let's see, you want a uxdxn. What I said is that all variations are along the y direction.

STUDENT: Oh, so this is not just for velocity, but for everything.

PROFESSOR: Yes, so I make an assumption about some particular form. So this is the reasoning. If these equations bring everything to equilibrium, I should be able to pick any initial condition and ask, how long does it take to come to equilibrium?

I pick this specific type of equation in which the only variations for all quantities are along the y direction. It's a non-equilibrium state. It's not a uniform state. Does it come relax to equilibrium? And the answer is no, it doesn't.

STUDENT: What other properties, other than velocity, is given [INAUDIBLE]?

PROFESSOR: Density and temperature. So these equations describe the variations of velocity, density, and temperature. And the statement is, if the system is to reach equilibrium, I should be able to start with any initial configuration of these three quantities that I want. And I see that after a while, it reaches a uniform state. Yes.

STUDENT: But if your initial conditions aren't exactly that, but you add a slight fluctuation, it is likely to grow, and it will eventually relax.

PROFESSOR: It turns out the answer is no. So I'm sort of approaching this problem from this more kind of hand-waving perspective. More correctly, what you can do is you can start with some initial condition that, let's say, is in equilibrium, and then do a perturbation, and ask whether the perturbation will eventually relax to 0 or not.

And let's in fact do that for another quantity, which is the sound mode. So let's imagine that we start with a totally nice, uniform state. There is zero velocity initially. The density is uniform. The temperature is uniform. And then what I do is I will start here. And I will start talking, creating a variation that propagates in this x direction.

So I generated a stream that is moving along the x direction. And presumably, as I move along the x direction, there is a velocity that changes with position and temperature. Now initially, I had the density. I said that was uniform. Once I make this sound, as I move along the x direction, and the air is flowing back and forth, what happens is that the density will vary from the uniform state. And the deviations from the uniform state I will indicate by mu.

Similarly, the third quantity, let's assume, will have a form such as this. And currently, I have written the most general form of variations that I can have along the x direction. You could do it in different directions. But let's say for simplicity, we stick with this. I haven't told you what mu theta and u are. So I have to see what they are consistent with the equations that we have up there.

One thing that I will assume is that these things are small perturbations around the uniform state. And uniform-- sorry, small perturbations typically means that what I intend to do is to do a linearized approximation. So basically, what I will do is I will essentially look at the linear version of these equations.

And again, maybe I didn't emphasize it before. Clearly these are nonlinear equations. Because let's say you have u grad u. It's the same nonlinearity that you have, let's say, in Navier-Stokes equation. Because you're transporting something

and moving along with the flow.

But when you do the linearization, then these operators that involve dt plus something like u-- I guess in this case, the only direction that is varying is x-- something like this of whatever quantity that I have, I can drop this nonlinear term.

Why? Because u is a perturbation around a uniform state. And gradients will pick up some perturbations around the uniform state. So essentially the linearization amounts to dropping these nonlinear components and some other things that I will linearizer also. Because all of these functions here, the derivatives act on product of n temperature over here. These are all nonlinear operations.

So let's linearize what we have. We have that Dt of the density-- I guess when I take the time derivative, I get n bar the time derivative of the quantity that I'm calling mu. And that's it. I don't need to worry about the convective part, the u dot grad part. That's second order.

On the right hand side, what do I have? I have ndu. Well, divergence of u is already the first variation. So for n, I will take its 0-th order term. So I have minus n bar dxux.

The equation for ux, really the only component that I have, is dt of ux. Actually, let's write down the equation for temperature. Let's look at this equation. So I have that dt acting on 3/2 kB times T. I will pick up T bar. And then I would have dt of theta.

What do I have on the right hand side? I have a derivative here. So everything else here I will evaluate at the 0-th order term, so n bar k T bar dxux. So I can see that I can certainly divide through n bar here. And one of my equations becomes dt of mu is minus dxux.

But from here, I see that dxux is also related once I divide by kT to 3/2 dt theta. And I know this to be true. And I seem to have an additional factor of 1 over n bar here. And so I made the mistake at some point, probably when I wrote this equation.

STUDENT: It's the third equation.

PROFESSOR: Yeah, so this should not be here. And that should not be here means that I probably

20

made a mistake here. So this should be a 1/n, sorry. There was indeed a 1/n / here. And there is no factor here.

So we have a relationship between the time derivatives of these variations in density and dx of ux. Fine, what does the equation for u tell us? It tells us that dt of ux is minus 1 over m n bar. Because of the derivative, I can set everything at the variation. And what do I have here? I have d by dx of n bar kB T bar. And if I look at the variations, I have 1 plus mu plus theta. The higher order terms I will forget. Yes.

STUDENT: Shouldn't that be plus 3/2 dt theta?

PROFESSOR: It should be plus, yes. There is a minus sign here. And that makes it plus. So the n bar we can take outside. This becomes minus kT over m at space variations of mu plus theta.

Now, what we do is that what I have here is information about the time derivatives of mu and theta. And here I have space derivatives. So what do I do? I basically apply an additionally dt here, which we'll apply here. And then we can apply it also here.

And then we know how dt of mu and dt of theta are related to dx of ux. The minus signs disappear. I have kB T bar divided by m. I have dt of mu is dxux. dt of theta is 2/3 dxux. So I will get 1 plus 2/3 dx squared of ux.

So the second derivative of ux in time is proportional to the second derivative of ux in space. So that's the standard wave equation. And the velocity that we have calculated for these sound waves is 5/3 kB the average T over m.

So that part is good. These equations tell me that if I create these disturbances, there are sound waves, and we know there are sound waves. And sound waves will propagate with some velocity that is related up to some factors to the average velocity of the gas particles.

But what is not good is that, according to this equation, if my waves, let's say, bounce off perfectly from the walls, they will last in this room forever. So you should still be hearing what I was saying last week and the week before. And clearly what

we are missing is the damping that is required.

So the statement is that all of these equations are fine. They capture a lot of the physics. But there is something important that is left out in that there are some modes-- and I describe two of them here-- that basically last forever, and don't come to equilibrium. But we said that the Boltzmann equation should eventually bring things to equilibrium.

So where did we go wrong? Well, we didn't solve the Boltzmann equation. We solved an approximation to the Boltzmann equation. So let's try to do better.

STUDENT: I'm sorry, but for the last equation, you took another derivative with respect to t.

PROFESSOR: Yes, I took a derivative with respect to t. And it noted that the derivative with respect to t of these quantities mu is related to derivative with respect to x or u. And there was one other derivative with respect to x already, making it two derivatives. So this is the kind of situation that we are facing. Yes.

STUDENT: Is the 5/3 k in any way related to the heat capacity ratio of [INAUDIBLE] gas?

PROFESSOR: Yes, that's right, yes. So there are lots of these things that are implicit in these questions. And actually, that 3/2 is the same thing as this 3/2. So you can trace a lot of these things to the Gaussian distribution. And they appear in cp versus cv and other things. Yes.

STUDENT: Just clarifying something-- this v is different from the mu in the top right?

PROFESSOR: Yes, this is v, and that's mu. This v is the velocity of the sound. So I defined this combination, the coefficient relating the second derivatives in time and space as the sound velocity. So let's maybe even-- we can call it vs. All right?

STUDENT: And how did you know that that is the [INAUDIBLE] oscillation, the solution that you got?

PROFESSOR: Because I know that the solution to dx squared anything is v squared-- sorry, v squared dx squared anything is dt squared anything, is phi is some function of x

minus vt. That is a pulse that moves with uniform velocity is a solution to this equation.

So we want to do better. And better becomes so-called first order solution. Now, the kind of equation that we are trying to solve at the top is something that its algebraic analog would be something like this-- 2 times x. It's a linear on the left hand side, is quadratic on the right hand side. Let's write it in this form-- except that the typical magnitude of one side is much larger than the other side, so let's say something like this.

So if I wanted to solve this equation, I would say that unless x is very close to 2, this 10 to the 6 will blow things up. So my x0 is 2. And that's what we have done. We've solved, essentially, the right hand side of the equation.

But I can get a better solution by taking 2 and saying there's a small variation to that that I want to calculate. And I substitute that into the original equation. On the left hand side, I will get 2 times 2, 1 plus epsilon. On the right hand side, I will get 10 to the sixth. And then essentially, I subtract 2 plus 2 epsilon squared from 5. What do I get? I will get 4 epsilon plus 4 epsilon squared.

Then I say that epsilon is small. So essentially, I linearize the right hand side. I forget about that. I say that I keep the epsilon here, because it's multiplying 10 to the 6. But the epsilon on the other side is multiplying nothing. So I forget that. So then I will have my epsilon to be roughly, I don't know, 2 times 2 divided by 4 times 10 to the 6. So I have gotten the correction to my first 0-th order solution to this first order.

Now we will do exactly the same thing, not for our algebraic equation, but for our Boltzmann equation. So for the Boltzmann equation, which was Lf is C of ff, we said that the right hand side is larger by a factor of 1 over tau x compared to the left hand side. And so what we did was we found a solution f0 that when we put in the collision integral, the answer was 0.

Now I want to have a better solution that I will call f1. Just like I did over there, I will

assume that f0 is added to a small function that I will call g. And then I substitute this equation, this thing, to the equation. So when I substitute, I will get L acting on f0 1 plus g.

Now, what did I do over here? On the left hand side, I ignored the first order term. Because I expect the first order term to be already small. And the left hand side is already small. So I will ignore this. And on the right hand side, I have to linearize. So I have to put f0 1 plus g, f0 1 plus g.

Essentially what I have to do is to go to the collisions that I have over here and write for this f0 1 plus g. There are four of such things. Now, the 0-th order term already cancels. Because f0 f0 was f0 f0 by the way that I constructed things.

And then I can pull out one factor of f0 out of the integration. So when I linearize this, what I will get is something like f0 that goes on the outside. I have the integral d2p2 d2b v2 minus v1. And then I have something like g of p1 plus g of p2 minus g of p1 prime minus g of p2 prime. So basically, what we have done is we have defined a linearized version of the collision operator that is now linear in this variable g.

Now in general, this is also still, although a linear operator, much simpler than the previous quadratic operator-- still has a lot of junk in it. So we are going to simply state that I will use a form of this linearized approximation that is simply g over tau, get rid of all of the integration. And this is called the single collision time approximation.

So having done that, what I have is that the L acting on f0 on the left hand side is minus f0 g over tau x on the right hand side. And so I can immediately solve for g. Because L is a first order derivative operator.

My g is minus tau x, the Liouville operator acting on f0-- sorry, log of f0. I divide it through by f0. So derivative of f0 divided by f0 is the derivative acting on log of f0.

So all I need to do is to essentially do the operations involved in taking these derivatives. Let's say we forget about the force. Because we are looking in the

middle of the box, acting on log of what I had written before. So what I have is log of n minus p minus mu squared over 2mkT. Remember, ukTn are all functions of position. So there will be derivatives involved here.

And I will just write down what the answer is. So the answer becomes minus tau x. You would have, once you do all of these derivatives and take advantage of the equations that you have written before-- so there's some lines of algebra involved. The final answer is going to be nm c alpha c beta minus-- I should really look at this. m over kT c alpha c beta minus delta alpha beta over 3 c squared u alpha beta, and then mc squared over 2kT minus 5/2 c alpha over T d alpha of T. Yes.

STUDENT: Sorry, what's the thing next to the c squared, something alpha beta?

PROFESSOR: Delta alpha beta, sorry. So there is a well-defined procedure-- it's kind of algebraically involved-- by which more or less in the same fashion that you can improve on the algebraic solution, get a better solution than the one that we had that now knows something about these relaxations. See, the 0-th order solution that we had knew nothing about tau x. We just set tau x to be very small, and set the right hand side to 0. And then nothing relaxed.

Now we have a better solution that involves explicitly tau x. And if we start with that, we'll find that we can get relaxation of all of these modes once we calculate p alpha beta and h alpha with this better solution. We can immediately, for example, see that this new solution will have terms that are odd. There is c cubed term here.

So when you're evaluating this average, you will no longer get 0. Heat has a chance to flow with this improved equation. And again, whereas before our pressure was diagonal because of these terms, we will have off diagonal terms that will allow us to relax the shear modes. And we'll do that next.