Review Problems

The enclosed exams (and solutions) from the previous years are intended to help you review the material.

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Note that the first parts of each problem are easier than its last parts. Therefore, make sure to proceed to the next problem when you get stuck.

You may find the following information helpful:

Physical Constants

Electron mass	$m_e \approx 9.1 \times 10^{-31} Kg$	Proton mass	$m_p \approx 1.7 \times 10^{-27} Kg$
Electron Charge	$e \approx 1.6 \times 10^{-19} C$	Planck's constant/2π	$\hbar \approx 1.1 \times 10^{-34} Js^1$
Speed of light	$c \approx 3.0 \times 10^8 ms^{-1}$	Stefan's constant	$\sigma \approx 5.7 \times 10^{-8} Wm^{-2} K^{-4}$
Boltzmann's constant	$k_B \approx 1.4 \times 10^{-23} JK^{-1}$	Avogadro's number	$N_0 \approx 6.0 \times 10^{23} mol^{-1}$

Conversion Factors

$$1 atm \equiv 1.0 \times 10^5 Nm^{-2} \qquad\qquad 1\mathring{A} \equiv 10^{-10} m \qquad\qquad 1 eV \equiv 1.1 \times 10^4 K$$

Thermodynamics

$$dE = TdS + dW \qquad\qquad \text{For a gas: } dW = -PdV \qquad\qquad \text{For a film: } dW = \sigma dA$$

Mathematical Formulas

$$\lim_{x \to \infty} \coth x = 1 + 2e^{-2x} + \mathcal{O}\left(e^{-4x}\right) \qquad\qquad \lim_{x \to 0} \coth x = \frac{1}{x} + \frac{x}{3} + \mathcal{O}\left(x^2\right)$$

$$\int_0^\infty dx \, x^n \, e^{-\alpha x} = \frac{n!}{\alpha^{n+1}} \qquad\qquad \left(\tfrac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^\infty dx \exp\left[-ikx - \frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2} \exp\left[-\frac{\sigma^2 k^2}{2}\right] \qquad\qquad \lim_{N \to \infty} \ln N! = N \ln N - N$$

$$\left\langle e^{-ikx} \right\rangle = \sum_{n=1}^\infty \frac{(-ik)^n}{n!} \left\langle x^n \right\rangle \qquad\qquad \ln \left\langle e^{-ikx} \right\rangle = \sum_{n=1}^\infty \frac{(-ik)^n}{n!} \left\langle x^n \right\rangle_c$$

$$f_m^\eta(z) = \frac{1}{(m-1)!} \int_0^\infty dx \frac{x^{m-1}}{z^{-1}e^x - \eta} = \sum_{\alpha=1}^\infty \eta^{\alpha+1} \frac{z^\alpha}{\alpha^m} \qquad\qquad \frac{df_m^\eta}{dz} = \frac{1}{z} f_{m-1}^\eta$$

$$\lim_{z \to \infty} f_m^-(z) = \frac{(\ln z)^m}{m!} \left[1 + \frac{\pi^2}{6} m(m-1)(\ln z)^{-2} + \cdots\right] \quad f_2^-(1) = \frac{\pi^2}{12} \quad f_4^-(1) = \frac{7\pi^4}{720}$$

$$\zeta_m \equiv f_m^+(1) \qquad \zeta_{3/2} \approx 2.612 \qquad \zeta_2 = \frac{\pi^2}{6} \qquad \zeta_{5/2} \approx 1.341 \qquad \zeta_3 \approx 1.202 \qquad \zeta_4 = \frac{\pi^4}{90}$$

1. *Exciton dissociation in a semiconductor:* By shining an intense laser beam on a semicon-ductor, one can create a metastable collection of electrons (charge $-e$, and effective mass m_e) and holes (charge $+e$, and effective mass m_h) in the bulk. The oppositely charged particles may pair up (as in a hydrogen atom) to form a gas of *excitons*, or they may dissociate into a plasma. We shall examine a much simplified model of this process.

(a) Calculate the free energy of a gas composed of N_e electrons and N_h holes, at temper-ature T, treating them as classical non-interacting particles of masses m_e and m_h.

(b) By pairing into an excition, the electron hole pair lowers its energy by ε. [The binding energy of a hydrogen-like exciton is $\varepsilon \approx me^4/(2\hbar^2\epsilon^2)$, where ϵ is the dielectric constant, and $m^{-1} = m_e^{-1} + m_h^{-1}$.] Calculate the free energy of a gas of N_p excitons, treating them as classical non-interacting particles of mass $m = m_e + m_h$.

(c) Calculate the chemical potentials μ_e, μ_h, and μ_p of the electron, hole, and exciton states, respectively.
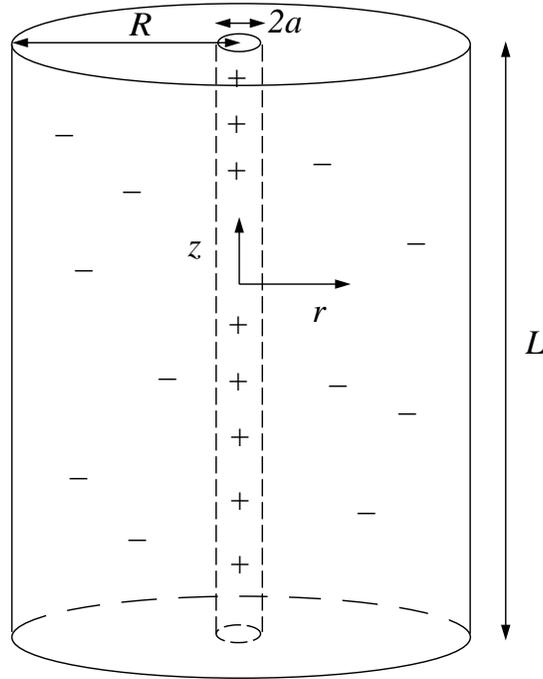
(d) Express the equilibrium condition between excitons and electron/holes in terms of their chemical potentials.

(e) At a high temperature T, find the density n_p of excitons, as a function of the total density of excitations $n \approx n_e + n_h$.

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2. *The Manning Transition:* When ionic polymers (polyelectrolytes) such as DNA are immersed in water, the negatively charged *counter-ions* go into solution, leaving behind a positively charged polymer. Because of the electrostatic repulsion of the charges left behind, the polymer stretches out into a cylinder of radius a, as illustrated in the figure. While thermal fluctuations tend to make the ions wander about in the solvent, electrostatic attractions favor their return and condensation on the polymer. If the number of counter-ions is N, they interact with the N positive charges left behind on the rod through the potential $U(r) = -2(Ne/L)\ln(r/L)$, where r is the radial coordinate in a cylindrical geometry. If we ignore the Coulomb repulsion between counter-ions, they can be described by the classical Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N}\left[\frac{p_i^2}{2m} + 2e^2 n \ln\left(\frac{r}{L}\right)\right],$$

where $n = N/L$.

(a) For a cylindrical container of radius R, calculate the canonical partition function Z in terms of temperature T, density n, and radii R and a.

(b) Calculate the probability distribution function $p(r)$ for the radial position of a counter-ion, and its first moment $\langle r \rangle$, the average radial position of a counter-ion.

(c) The behavior of the results calculated above in the limit $R \gg a$ is very different at high and low temperatures. Identify the transition temperature, and characterize the nature of the two phases. In particular, how does $\langle r \rangle$ depend on R and a in each case?

(d) Calculate the pressure exerted by the counter-ions on the wall of the container, at $r = R$, in the limit $R \gg a$, at all temperatures.

(e) The character of the transition examined in part (d) is modified if the Coulomb interactions between counter-ions are taken into account. An approximate approach to the interacting problem is to allow a fraction N_1 of counter-ions to condense along the polymer rod, while the remaining $N_2 = N - N_1$ fluctuate in the solvent. The free counter-ions are again treated as non-interacting particles, governed by the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} + 2e^2 n_2 \ln \left(\frac{r}{L} \right) \right],$$

where $n_2 = N_2/L$. *Guess* the equilibrium number of non-interacting ions, N_2^*, and justify your guess by discussing the response of the system to slight deviations from N_2^*. (This is a qualitative question for which no new calculations are needed.)

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3. *Bose gas in d dimensions:* Consider a gas of non-interacting (spinless) bosons with an energy spectrum $\epsilon = p^2/2m$, contained in a box of "volume" $V = L^d$ in d dimensions.

(a) Calculate the grand potential $\mathcal{G} = -k_B T \ln \mathcal{Q}$, and the density $n = N/V$, at a chemical potential μ. Express your answers in terms of d and $f_m^+(z)$, where $z = e^{\beta \mu}$, and

$$f_m^+(z) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{x^{m-1}}{z^{-1} e^x - 1} dx.$$

(Hint: Use integration by parts on the expression for $\ln \mathcal{Q}$.)

(b) Calculate the ratio PV/E, and compare it to the classical value.

(c) Find the critical temperature, $T_c(n)$, for Bose-Einstein condensation.

(d) Calculate the heat capacity $C(T)$ for $T < T_c(n)$.

(e) Sketch the heat capacity at all temperatures.

(f) Find the ratio, $C_{\max}/C(T \to \infty)$, of the maximum heat capacity to its classical limit, and evaluate it in $d = 3$

(g) How does the above calculated ratio behave as $d \to 2$? In what dimensions are your results valid? Explain.

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1. *Electron Magnetism:* The conduction electrons in a metal can be treated as a gas of fermions of spin 1/2 (with up/down degeneracy), and density $n = N/V$.

(a) Ignoring the interactions between electrons, describe (in words) their ground state. Calculate the fermi wave number k_F, and the ground-state energy density E_0/V in terms of the density n.

Electrons also interact *via* the Coulomb repulsion, which favors a wave function which is antisymmetric in position space, thus keeping them apart. Because of the full (position *and* spin) antisymmetry of fermionic wave functions, this interaction may be described as an effective spin-spin coupling which favors states with parallel spins. In a simple approximation, the effect of this interaction is represented by adding a potential

$$U = \alpha \frac{N_+ N_-}{V},$$

to the Hamiltonian, where N_+ and $N_- = N - N_+$ are the numbers of electrons with up and down spins, and V is the volume. (The parameter α is related to the scattering length a by $\alpha = 4\pi\hbar^2 a/m$.) We would like to find out if the unmagnetized gas with $N_+ = N_- = N/2$ still minimizes the energy, or if the gas is spontaneously magnetized.

(b) Express the modified Fermi wave numbers k_{F+} and k_{F-}, in terms of the densities $n_+ = N_+/V$ and $n_- = N_-/V$.

(c) Assuming small deviations $n_+ = n/2 + \delta$ and $n_- = n/2 - \delta$ from the symmetric state, calculate the change in the kinetic energy of the system to second order in δ.

(d) Express the spin-spin interaction density in terms of δ. Find the critical value of α_c, such that for $\alpha > \alpha_c$ the electron gas can lower its total energy by spontaneously developing a magnetization. (This is known as the *Stoner instability*.)

(e) Explain qualitatively, and sketch the behavior of the spontaneous magnetization as a function of α.

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2. *Boson magnetism:* Consider a gas of non-interacting spin 1 bosons, each subject to a Hamiltonian

$$\mathcal{H}_1(\vec{p}, s_z) = \frac{\vec{p}^2}{2m} - \mu_0 s_z B \quad,$$

where $\mu_0 = e\hbar/mc$, and s_z takes *three* possible values of (-1, 0, +1). (The orbital effect, $\vec{p} \to \vec{p} - e\vec{A}$, has been ignored.)

(a) In a grand canonical ensemble of chemical potential μ, what are the average occupation numbers $\left\{ \langle n_+(\vec{k})\rangle, \langle n_0(\vec{k})\rangle, \langle n_-(\vec{k})\rangle \right\}$, of one-particle states of wavenumber $\vec{k} = \vec{p}/\hbar$?

(b) Calculate the average total numbers $\{N_+, N_0, N_-\}$, of bosons with the three possible values of s_z in terms of the functions $f_m^+(z)$.

(c) Write down the expression for the magnetization $M(T, \mu) = \mu_0(N_+ - N_-)$, and by expanding the result for small B find the *zero field susceptibility* $\chi(T, \mu) = \partial M/\partial B|_{B=0}$.

To find the behavior of $\chi(T, n)$, where $n = N/V$ is the total density, proceed as follows:

(d) For $B = 0$, find the high temperature expansion for $z(\beta, n) = e^{\beta\mu}$, correct to second order in n. Hence obtain the first correction from quantum statistics to $\chi(T, n)$ at high temperatures.

(e) Find the temperature $T_c(n, B = 0)$, of Bose-Einstein condensation. What happens to $\chi(T, n)$ on approaching $T_c(n)$ from the high temperature side?

(f) What is the chemical potential μ for $T < T_c(n)$, at a small but finite value of B? Which one-particle state has a macroscopic occupation number?

(g) Using the result in (f), find the spontaneous magnetization,

$$\overline{M}(T, n) = \lim_{B \to 0} M(T, n, B).$$

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3. *The virial theorem* is a consequence of the invariance of the phase space for a system of N (classical or quantum) particles under canonical transformations, such as a change of scale. In the following, consider N particles with coordinates $\{\vec{q}_i\}$, and conjugate momenta $\{\vec{p}_i\}$ (with $i = 1, \cdots, N$), and subject to a Hamiltonian $\mathcal{H}(\{\vec{p}_i\}, \{\vec{q}_i\})$.

(a) *Classical version*: Write down the expression for classical partition function, $Z \equiv Z[\mathcal{H}]$. Show that it is invariant under the rescaling $\vec{q}_1 \to \lambda\vec{q}_1$, $\vec{p}_1 \to \vec{p}_1/\lambda$ of a pair of conjugate variables, i.e. $Z[\mathcal{H}_\lambda]$ is independent of λ, where $\mathcal{H}_\lambda$ is the Hamiltonian obtained after the above rescaling.

(b) *Quantum mechanical version*: Write down the expression for quantum partition function. Show that it is also invariant under the rescalings $\vec{q}_1 \to \lambda\vec{q}_1$, $\vec{p}_1 \to \vec{p}_1/\lambda$, where $\vec{p}_i$ and $\vec{q}_i$ are now quantum mechanical operators. (Hint: start with the time-independent Schrödinger equation.)

(c) Now assume a Hamiltonian of the form

$$\mathcal{H} = \sum_i \frac{\vec{p}_i^{\ 2}}{2m} + V(\{\vec{q}_i\}).$$

Use the result that $Z[\mathcal{H}_\lambda]$ is independent of λ to prove the *virial* relation

$$\left\langle \frac{\vec{p}_1^{\ 2}}{m} \right\rangle = \left\langle \frac{\partial V}{\partial \vec{q}_1} \cdot \vec{q}_1 \right\rangle,$$

where the brackets denote thermal averages. (You may formulate your answer in the classical language, as a possible quantum derivation is similar.)

(d) The above relation is sometimes used to estimate the mass of distant galaxies. The stars on the outer boundary of the G-8.333 galaxy have been measured to move with velocity $v \approx 200$ km/s. Give a numerical estimate of the ratio of the G-8.333's mass to its size.

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1. *Freezing of He³:* At low temperatures He³ can be converted from liquid to solid by application of pressure. A peculiar feature of its phase boundary is that $(dP/dT)_{\text{melting}}$ is negative at temperatures below $0.3\,^oK$ $[(dP/dT)_m \approx -30\text{atm}\,^oK^{-1}$ at $T \approx 0.1\,^oK]$. We will use a simple model of liquid and solid phases of He³ to account for this feature.

(a) In the solid phase, the He³ atoms form a crystal lattice. Each atom has nuclear spin of $1/2$. Ignoring the interaction between spins, what is the entropy per particle s_s, due to the spin degrees of freedom?

(b) Liquid He³ is modelled as an ideal Fermi gas, with a volume of $46\mathring{A}^3$ per atom. What is its Fermi temperature T_F, in degrees Kelvin?

(c) How does the heat capacity of liquid He³ behave at low temperatures? Write down an expression for C_V in terms of N, T, k_B, T_F, *up to a numerical constant*, that is valid for $T \ll T_F$.

(d) Using the result in (c), calculate the entropy per particle s_ℓ, in the liquid at low temperatures. For $T \ll T_F$, which phase (solid or liquid) has the higher entropy?

(e) By equating chemical potentials, or by any other technique, prove the Clausius–Clapeyron equation $(dP/dT)_{\text{melting}} = (s_\ell - s_s)/(v_\ell - v_s)$, where v_ℓ and v_s are the volumes per particle in the liquid and solid phases respectively.

(f) It is found experimentally that $v_\ell - v_s = 3\mathring{A}^3$ per atom. Using this information, plus the results obtained in previous parts, estimate $(dP/dT)_{\text{melting}}$ at $T \ll T_F$.
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2. *Non-interacting bosons:* Consider a grand canonical ensemble of non-interacting *bosons* with chemical potential μ. The one–particle states are labelled by a wavevector $\vec{q}$, and have energies $\mathcal{E}(\vec{q})$.

(a) What is the joint probability $P(\{n_{\vec{q}}\})$, of finding a set of occupation numbers $\{n_{\vec{q}}\}$, of the one–particle states, in terms of the fugacities $z_{\vec{q}} \equiv \exp[\beta(\mu - \mathcal{E}(\vec{q}))]$?

(b) For a particular $\vec{q}$, calculate the characteristic function $\langle \exp[ikn_{\vec{q}}]\rangle$.
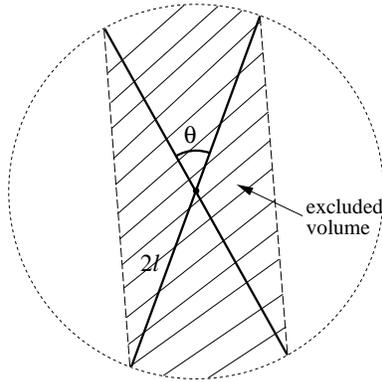
(c) Using the result of part (b), **or otherwise**, give expressions for the mean and variance of $n_{\vec{q}}$. occupation number $\langle n_{\vec{q}}\rangle$.

(d) Express the variance in part (c) in terms of the mean occupation number $\langle n_{\vec{q}}\rangle$.

(e) Express your answer to part (a) in terms of the occupation numbers $\{\langle n_{\vec{q}}\rangle\}$.

(f) Calculate the entropy of the probability distribution for bosons, in terms of $\{\langle n_{\vec{q}}\rangle\}$, and comment on its zero temperature limit.
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3. *Hard rods:* A collection of N asymmetric molecules in two dimensions may be modeled as a gas of rods, each of length $2l$ and lying in a plane. A rod can move by translation of its center of mass and rotation about latter, as long as it does not encounter another rod. Without treating the hard-core interaction exactly, we can incorporate it approximately by assuming that the rotational motion of each rod is restricted (by the other rods) to an angle θ, which in turn introduces an excluded volume $\Omega(\theta)$ (associated with each rod). The value of θ is then calculated self consistently by maximizing the entropy at a given density $n = N/V$, where V is the total accessible area.

(a) Write down the entropy of such a collection of rods in terms of N, n, Ω, and $A(\theta)$, the entropy associated to the rotational freedom of a *single* rod. (You may ignore the momentum contributions throughout, and consider the large N limit.)

(b) Extremizing the entropy as a function of θ, relate the density to Ω, A, and their derivatives Ω', A'; express your result in the form $n = f(\Omega, A, \Omega', A')$.

(c) Express the excluded volume Ω in terms of θ and sketch f as a function of $\theta \in [0, \pi]$, assuming $A \propto \theta$.

(d) Describe the equilibrium state at high densities. Can you identify a phase transition as the density is decreased? Draw the corresponding critical density n_c on your sketch. What is the critical angle θ_c at the transition? You don't need to calculate θ_c explicitly, but give an (implicit) relation defining it. What value does θ adopt at $n < n_c$?

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1. *Helium 4:* [4]He at low temperatures can be converted from liquid to solid by application of pressure. An interesting feature of the phase boundary is that the melting pressure is reduced slightly from its $T = 0$K value, by approximately 20Nm^{-2} at its minimum at $T = 0.8$K. We will use a simple model of liquid and solid phases of [4]He to account for this feature.

(a) By equating chemical potentials, or by any other technique, prove the Clausius–Clapeyron equation $(dP/dT)_{\text{melting}} = (s_\ell - s_s)/(v_\ell - v_s)$, where (v_ℓ, s_ℓ) and (v_s, s_s) are the volumes and entropies per atom in the liquid and solid phases respectively.

(b) The important excitations in liquid [4]He at $T < 1°K$ are phonons of velocity c. Calculate the contribution of these modes to the heat capacity per particle C_V^ℓ/N, of the liquid.

(c) Calculate the low temperature heat capacity per particle C_V^s/N, of solid [4]He in terms of longitudinal and transverse sound velocities c_L, and c_T.

(d) Using the above results calculate the entropy difference $(s_\ell - s_s)$, assuming a single sound velocity $c \approx c_L \approx c_T$, and approximately equal volumes per particle $v_\ell \approx v_s \approx v$. Which phase (solid or liquid) has the higher entropy?

(e) Assuming a small (temperature independent) volume difference $\delta v = v_\ell - v_s$, calculate the form of the melting curve. To explain the anomaly described at the beginning, which phase (solid or liquid) must have the higher density?

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2. *Surfactant Condensation:* N surfactant molecules are added to the surface of water over an area A. They are subject to a Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \frac{\vec{p_i}^{\,2}}{2m} + \frac{1}{2}\sum_{i,j}\mathcal{V}(\vec{r}_i - \vec{r}_j),$$

where $\vec{r}_i$ and $\vec{p}_i$ are two dimensional vectors indicating the position and momentum of particle i.

(a) Write down the expression for the partition function $Z(N, T, A)$ in terms of integrals over $\vec{r}_i$ and $\vec{p}_i$, and perform the integrals over the momenta.

The inter–particle potential $\mathcal{V}(\vec{r})$ is infinite for separations $|\vec{r}| < a$, and attractive for $|\vec{r}| > a$ such that $\int_a^\infty 2\pi r dr \mathcal{V}(r) = -u_0$.

(b) Estimate the total non–excluded area available in the positional phase space of the system of N particles.

(c) Estimate the total *potential* energy of the system, *assuming a constant density* $n = N/A$. Assuming this potential energy for all configurations allowed in the previous part, write down an approximation for Z.

(d) The surface tension of water without surfactants is σ_0, approximately independent of temperature. Calculate the surface tension $\sigma(n, T)$ in the presence of surfactants.

(e) Show that below a certain temperature, T_c, the expression for σ is manifestly incorrect. What do you think happens at low temperatures?

(f) Compute the heat capacities, C_A and write down an expression for C_σ without explicit evaluation, due to thesurfactants.

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3. *Dirac Fermions* are non-interacting particles of spin $1/2$. The one-particle states come in pairs of positive and negative energies,

$$\mathcal{E}_\pm(\vec{k}) = \pm\sqrt{m^2c^4 + \hbar^2 k^2 c^2} \quad ,$$

independent of spin.

(a) For *any* fermionic system of chemical potential μ, show that the probability of finding an occupied state of energy $\mu + \delta$ is the same as that of finding an unoccupied state of energy $\mu - \delta$. (δ is any constant energy.)

(b) At zero temperature all negative energy Dirac states are occupied and all positive energy ones are empty, i.e. $\mu(T = 0) = 0$. Using the result in (a) find the chemical potential at finite temperatures T.

(c) Show that the mean excitation energy of this system at finite temperature satisfies

$$E(T) - E(0) = 4V \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\mathcal{E}_+(\vec{k})}{\exp\left(\beta\mathcal{E}_+(\vec{k})\right) + 1} \quad .$$

(d) Evaluate the integral in part (c) for *massless Dirac particles* (i.e. for $m = 0$).

(e) Calculate the heat capacity, C_V, of such massless Dirac particles.

(f) Describe the qualitative dependence of the heat capacity at low temperature if the particles are massive.

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1. *Neutron star core:* Professor Rajagopal's group has proposed that a new phase of QCD matter may exist in the core of neutron stars. This phase can be viewed as a condensate of quarks in which the low energy excitations are approximately

$$\mathcal{E}(\vec{k})_{\pm} =_{\pm} \hbar^2 \frac{\left(|\vec{k}| - k_F\right)^2}{2M} \quad .$$

The excitations are fermionic, with a degeneracy of $g = 2$ from spin.

(a) At zero temperature all negative energy states are occupied and all positive energy ones are empty, i.e. $\mu(T = 0) = 0$. By relating occupation numbers of states of energies $\mu + \delta$ and $\mu - \delta$, or otherwise, find the chemical potential at finite temperatures T.

(b) Assuming a constant density of states near $k = k_F$, i.e. setting $d^3k \approx 4\pi k_F^2 dq$ with $q = |\vec{k}| - k_F$, show that the mean excitation energy of this system at finite temperature is

$$E(T) - E(0) \approx 2gV\frac{k_F^2}{\pi^2} \int_0^\infty dq \frac{\mathcal{E}_+(q)}{\exp\left(\beta\mathcal{E}_+(q)\right) + 1} \quad .$$

(c) Give a closed form answer for the excitation energy by evaluating the above integral.

(d) Calculate the heat capacity, C_V, of this system, and comment on its behavior at low temperature.

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2. *Critical point behavior:* The pressure P of a gas is related to its density $n = N/V$, and temperature T by the truncated expansion

$$P = k_B T n - \frac{b}{2}n^2 + \frac{c}{6}n^3 \quad ,$$

where b and c are assumed to be *positive* temperature independent constants.

(a) Locate the critical temperature T_c below which this equation must be invalid, and the corresponding density n_c and pressure P_c of the critical point. Hence find the ratio $k_B T_c n_c / P_c$.

(b) Calculate the isothermal compressibility $\kappa_T = -\frac{1}{V}\frac{\partial V}{\partial P}\big|_T$, and sketch its behavior as a function of T for $n = n_c$.

(c) On the critical isotherm give an expression for $(P - P_c)$ as a function of $(n - n_c)$.

(d) The instability in the isotherms for $T < T_c$ is avoided by phase separation into a liquid of density n_+ and gas of density n_-. For temperatures close to T_c, these densities behave as $n_\pm \approx n_c(1 \pm \delta)$. Using a Maxwell construction, or otherwise, find an implicit equation

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for $\delta(T)$, and indicate its behavior for $(T_c - T) \to 0$. (Hint: Along an isotherm, variations of chemical potential obey $d\mu = dP/n$.)

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3. *Relativistic Bose gas in d dimensions:* Consider a gas of non-interacting (spinless) bosons with energy $\epsilon = c\,|\vec{p}|$, contained in a box of "volume" $V = L^d$ in d dimensions.

(a) Calculate the grand potential $\mathcal{G} = -k_{\mathrm{B}} T \ln \mathcal{Q}$, and the density $n = N/V$, at a chemical potential μ. Express your answers in terms of d and $f_m^+(z)$, where $z = e^{\beta\mu}$, and

$$ f_m^+(z) = \frac{1}{(m-1)!} \int_0^\infty \frac{x^{m-1}}{z^{-1}e^x - 1} dx. $$

(Hint: Use integration by parts on the expression for $\ln \mathcal{Q}$.)

(b) Calculate the gas pressure P, its energy E, and compare the ratio $E/(PV)$ to the classical value.

(c) Find the critical temperature, $T_c(n)$, for Bose-Einstein condensation, indicating the dimensions where there is a transition.

(d) What is the temperature dependence of the heat capacity $C(T)$ for $T < T_c(n)$?

(e) Evaluate the dimensionless heat capacity $C(T)/(Nk_B)$ at the critical temperature $T = T_c$, and compare its value to the classical (high temperature) limit.

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1. *Graphene* is a single sheet of carbon atoms bonded into a *two dimensional* hexagonal lattice. It can be obtained by exfoliation (repeated peeling) of graphite. The band structure of graphene is such that the single particles excitations behave as relativistic Dirac *fermions*, with a spectrum that at low energies can be approximated by

$$\mathcal{E}_{\pm}(\vec{k}) = \pm\hbar v \left|\vec{k}\right| \quad .$$

There is spin degeneracy of $g = 2$, and $v \approx 10^6 \mathrm{ms}^{-1}$. Recent experiments on unusual transport properties of graphene were reported in *Nature* **438**, 197 (2005). In this problem, you shall calculate the heat capacity of this material.

(a) If at zero temperature all negative energy states are occupied and all positive energy ones are empty, find the chemical potential $\mu(T)$.

(b) Show that the mean excitation energy of this system at finite temperature satisfies

$$E(T) - E(0) = 4A \int \frac{d^2\vec{k}}{(2\pi)^2} \frac{\mathcal{E}_+(\vec{k})}{\exp\left(\beta\mathcal{E}_+(\vec{k})\right) + 1} \quad .$$

(c) Give a closed form answer for the excitation energy by evaluating the above integral.

(d) Calculate the heat capacity, C_V, of such massless Dirac particles.

(e) Explain qualitatively the contribution of phonons (lattice vibrations) to the heat capacity of graphene. The typical sound velocity in graphite is of the order of $2 \times 10^4 \mathrm{ms}^{-1}$. Is the low temperature heat capacity of graphene controlled by phonon or electron contributions?

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2. *Quantum Coulomb gas:* Consider a *quantum* system of N positive, and N negative charged relativistic particles in box of volume $V = L^3$. The Hamiltonian is

$$\mathcal{H} = \sum_{i=1}^{2N} c|\vec{p}_i| + \sum_{i<j}^{2N} \frac{e_i e_j}{|\vec{r}_i - \vec{r}_j|} \quad ,$$

where $e_i = +e_0$ for $i = 1, \cdots N$, and $e_i = -e_0$ for $i = N+1, \cdots 2N$, denote the charges of the particles; $\{\vec{r}_i\}$ and $\{\vec{p}_i\}$ their coordinates and momenta respectively. While this is too complicated a system to solve, we can nonetheless obtain some exact results.

(a) Write down the Schrödinger equation for the eigenvalues $\varepsilon_n(L)$, and (in coordinate space) eigenfunctions $\Psi_n(\{\vec{r}_i\})$. State the constraints imposed on $\Psi_n(\{\vec{r}_i\})$ if the particles are bosons or fermions?

(b) By a change of scale $\vec{r}_i{}' = \vec{r}_i/L$, show that the eigenvalues satisfy a scaling relation $\varepsilon_n(L) = \varepsilon_n(1)/L$.

(c) Using the formal expression for the partition function $Z(N, V, T)$, in terms of the eigenvalues $\{\varepsilon_n(L)\}$, show that Z does not depend on T and V separately, but only on a specific scaling combination of them.

(d) Relate the energy E, and pressure P of the gas to variations of the partition function. Prove the exact result $E = 3PV$.

(e) The Coulomb interaction between charges in in d-dimensional space falls off with separation as $e_i e_j / |\vec{r}_i - \vec{r}_j|^{d-2}$. (In $d = 2$ there is a logarithmic interaction.) In what dimension d can you construct an exact relation between E and P for *non-relativistic* particles (kinetic energy $\sum_i \vec{p}_i^2/2m$)? What is the corresponding exact relation between energy and pressure?

(f) Why are the above 'exact' scaling laws not expected to hold in dense (liquid or solid) Coulomb mixtures?

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3. *Non-interacting Fermions:* Consider a grand canonical ensemble of non-interacting *fermions* with chemical potential μ. The one–particle states are labelled by a wavevector $\vec{k}$, and have energies $\mathcal{E}(\vec{k})$.

(a) What is the joint probability $P(\{n_{\vec{k}}\})$, of finding a set of occupation numbers $\{n_{\vec{k}}\}$, of the one–particle states?

(b) Express your answer to part (a) in terms of the average occupation numbers $\left\{ \langle n_{\vec{k}} \rangle_- \right\}$.

(c) A random variable has a set of ℓ discrete outcomes with probabilities p_n, where $n = 1, 2, \cdots, \ell$. What is the entropy of this probability distribution? What is the maximum possible entropy?

(d) Calculate the entropy of the probability distribution for fermion occupation numbers in part (b), and comment on its zero temperature limit.

(e) Calculate the variance of the total number of particles $\langle N^2 \rangle_c$, and comment on its zero temperature behavior.

(f) The number fluctuations of a gas is related to its compressibility κ_T, and number density $n = N/V$, by

$$\langle N^2 \rangle_c = Nnk_BT\kappa_T \quad .$$

Give a *numerical estimate* of the compressibility of the fermi gas in a metal at $T = 0$ in units of $\mathring{A}^3 eV^{-1}$.

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1. *Freezing of He⁴:* At low temperatures He^4 can be converted from liquid to solid by application of pressure. An interesting feature of the phase boundary is that the melting pressure is reduced slightly from its $T = 0K$ value, by approximately 20Nm^{-2} at its minimum at $T = 0.8K$. We will use a simple model of liquid and solid phases of ^{4}He to account for this feature.

(a) By equating chemical potentials, or by any other technique, prove the Clausius–Clapeyron equation $(dP/dT)_{\text{melting}} = (s_\ell - s_s)/(v_\ell - v_s)$, where (v_ℓ, s_ℓ) and (v_s, s_s) are the volumes and entropies per atom in the liquid and solid phases respectively.

(b) The important excitations in liquid ^{4}He at $T < 1$K are phonons of velocity c. Calculate the contribution of these modes to the heat capacity per particle C_V^ℓ/N, of the liquid.

(c) Calculate the low temperature heat capacity per particle C_V^s/N, of solid ^{4}He in terms of longitudinal and transverse sound velocities c_L, and c_T.

(d) Using the above results calculate the entropy difference $(s_\ell - s_s)$, assuming a single sound velocity $c \approx c_L \approx c_T$, and approximately equal volumes per particle $v_\ell \approx v_s \approx v$. Which phase (solid or liquid) has the higher entropy?

(e) Assuming a small (temperature independent) volume difference $\delta v = v_\ell - v_s$, calculate the form of the melting curve. To explain the anomaly described at the beginning, which phase (solid or liquid) must have the higher density?
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2. *Squeezed chain:* A rubber band is modeled as a single chain of N *massless* links of fixed length a. The chain is placed inside a narrow tube that restricts each link to point parallel or anti-parallel to the tube.

(a) Ignoring any interactions amongst the links, give the number of configurations of the chain, $\Omega(L, N)$, where L is the end–to–end length of the chain.

(b) Use Stirling's approximation to express the result for entropy as a function of N and $x = L/(Na)$.

   The tube is now uniformly squeezed such that stretched configurations are energetically favored. For this problem, assume a simplified form of this energy that depends inversely on the local link density, such that the energy of a uniformly stretched state is

$$E(L, N) = -\frac{\sigma}{2} L \times \frac{L}{N} = -\frac{\sigma a^2}{2} N x^2.$$

(Use this formula for energy for all states of given L and N in the remainder of this problem.)

(c) Calculate the free energy $F(T, L, N)$, and the force $J(T, L, N)$ acting on the end points of the chain. (The work done on expanding the chain is $dW = JdL$.)

(d) Sketch the isotherms $J(x, T)$ at high and low temperatures, and identify the temperature T_c when the behavior changes.

(e) What is the condition for stability of the chain? What portion of the above isotherms are inherently unstable.

(f) Using the form for J obtained in part (c), find an expression for the unforced $(J = 0)$ chain length that is valid as T approaches T_c from below.

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3. *Graphene bilayer:* The layers of graphite can be peeled apart through different exfoliation processes. Many such processes generate single sheets of carbon atoms, as well as bilayers in which the two sheets are weakly coupled. The hexagonal lattice of the single layer graphene, leads to a band structure that at low energies can be approximated by $\mathcal{E}_\pm^{1 \text{ layer}}(\vec{k}) = \pm t_\parallel (ak)$, as in relativistic Dirac *fermions*. (Here $k = |\vec{k}|$, a is a lattice spacing, and $t_\parallel$ is a typical in-plane hopping energy.) A weak hopping energy $t_\perp$ between the two sheets of the bilayer modifies the low energy excitations drastically, to

$$\mathcal{E}_\pm^{\text{bilayer}}(\vec{k}) = \pm \frac{t_\parallel^2}{2t_\perp}(ka)^2 \quad ,$$

i.e. resembling massive Dirac fermions. In addition to the spin degeneracy, there are two branches of such excitations per unit cell, for an overall degeneracy of $g = 4$.

(a) For the undoped material with one electron per site, at zero temperature all negative energy states are occupied and all positive energy ones are empty. Find the chemical potential $\mu(T)$.

(b) Show that the mean excitation energy of this system at finite temperature satisfies

$$E(T) - E(0) = 2gA \int \frac{d^2\vec{k}}{(2\pi)^2} \frac{\mathcal{E}_+(\vec{k})}{\exp\left(\beta \mathcal{E}_+(\vec{k})\right) + 1} \quad .$$

(c) Give a closed form answer for the excitation energy *of the bilayer* by evaluating the above integral.

(d) Calculate the heat capacity, C_A, of such massive Dirac particles.

(e) A sample contains an equal proportion of single and bilayers. Estimate (in terms of the hopping energies) the temperature below which the electronic heat capacity is dominated by the bilayers.

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