8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

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Lecture 2

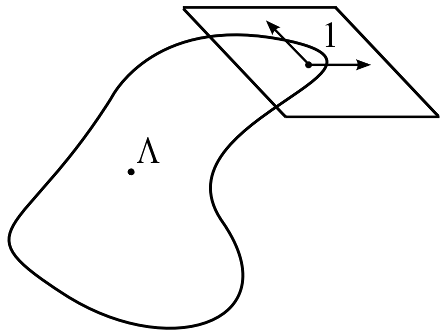

Figure 1: An element Λ of the manifold of the Lie group G, and the Lie algebra $\mathfrak{g}$ as the tangent space of the identity element.

Some facts about Lie groups and Lie algebras:

1. Different Lie groups can have the same Lie algebra. The Lie algebra determines the Lie group up to discrete choices of global structure. For example, $SU(2) = S^3$, $SO(3) = S^3/\mathbb{Z}_2$.

2. An **invariant subalgebra** is a subset of a Lie algebra $\mathfrak{g}' \subset \mathfrak{g}$ which is closed under the action of $\mathfrak{g}$. That is, $[\mathfrak{g}, \mathfrak{g}'] \subset \mathfrak{g}'$. A **simple** Lie algebra is a Lie algebra which does not contain invariant subalgebras and which is not Abelian. The complex simple Lie algebras are completely classified: $\mathfrak{su}(n)$, $\mathfrak{so}(2n)$, $\mathfrak{so}(2n+1)$, $\mathfrak{sp}(n)$, $E_{6,7,8}$, F_4 and G_2 are the only possibilities.

3. For a compact Lie group, it is always possible to choose a basis of T_a so that $f_{abc} = f^a_{\ bc}$ is truly antisymmetric (there is no distinction between upper and lower indices). All internal symmetry groups are compact. For example, $SU(n)$ (the set of $n \times n$ unitary matrices):
$$U = \exp\left[i\Lambda^a T_a\right], \ a = 1, \ldots, n^2 - 1, \tag{1}$$
where
$$\mathrm{Tr}(T_a) = 0, \ (T_a)^\dagger = T_a, \tag{2}$$
that is, the generators are hermitian and traceless, and hence we can choose
$$[T_a, T_b] = i f_{abc} T_c, \tag{3}$$
where the f_{abc} are fully antisymmetric.

4. Physically, for example considering $\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$ with an $SU(n)$ symmetry, we find a set of associated Noether charges $\hat{Q}_a$, $a = 1, \ldots n^2 - 1$, satisfying the Lie algebra commutation relations, $\left[\hat{Q}_a, \hat{Q}_b\right] = i f_{abc} \hat{Q}_c$. Then the transformations on Ψ are generated by the Noether charges:
$$\hat{U} = \exp\left[\Lambda^a \hat{Q}_a\right], \tag{4}$$
where $\left[\epsilon^a \hat{Q}_a, \Psi\right] = \epsilon^a T_a \Psi$. That is,
$$\hat{U}\Psi\hat{U}^\dagger = U\Psi, \tag{5}$$
where U is given in (1). This is checked explicitly for $SU(2)$ in the problem set.

1.2: THE GAUGE PRINCIPLE (QUANTUM ELECTRODYNAMICS REVISITED)

Referring back to the $U(1)$ invariant Lagrangian we studied in lecture 1:

$$\mathscr{L} = -i\overline{\psi}(\gamma^\mu \partial_\mu - m)\psi, \tag{6}$$

which is symmetric under $\psi(t,\vec{x}) \longrightarrow e^{i\alpha}\psi(t,\vec{x})$, we note that for the Lagrangian to be symmetric, it is necessary that α is not position-dependent. That is, all spacetime points transform in the same way. The transformation is no longer a symmetry for general $\alpha = \alpha(x)$, that is, if we allow different phase rotations at different spacetime points. The mass term is invariant under these more general transformations. The kinetic term, however, transforms as

$$\partial_\mu \Psi \to \partial_\mu(e^{i\alpha(x)}\Psi(x)) = e^{i\alpha(x)}\partial_\mu(\Psi(x)) + i\partial_\mu(\alpha(x))e^{i\alpha(x)}\Psi(x), \tag{7}$$

where we have kept the x-dependence explicit. The second term is the problem. We want to construct a theory (i.e. a Lagrangian) which is invariant for a general $\alpha(x)$, that is, a theory with a local $U(1)$ symmetry. The answer involves the introduction of a new vector field, and leads to quantum electrodynamics, as studied in 8.323. This example, in fact, embodies a deep principle: the principle of gauge invariance. As we will discuss,

$$\begin{aligned} \text{Local symmetries} &\Rightarrow \text{Interactions},\\ \text{Local } U(1)\text{symmetry} &\Rightarrow \text{Electromagnetic interaction},\\ \text{Local } U(n)\text{symmetries} &\Rightarrow \text{non-Abelian gauge interactions}. \end{aligned}$$

To illustrate this principle, we will now "rederive" Quantum Electrodynamics from the requirement of local $U(1)$ symmetry. We would like to construct a theory which is invariant under

$$\psi(t,\vec{x}) \longrightarrow e^{i\alpha(x)}\psi(t,\vec{x}), \tag{8}$$

for general $\alpha(x)$, also called a gauge transformation. An immediate consequence of (8) is that the ordinary derivative loses its physical meaning. Consider the derivative along some direction n^μ :

$$n^\mu \partial_\mu \psi = \lim_{\epsilon \to 0} \frac{\psi(x+\epsilon n) - \psi(x)}{\epsilon}. \tag{9}$$

If we can rotate $\psi(x+\epsilon n)$ and $\psi(x)$ independently, (9) does not have a definite meaning, as can be seen from the last term in (7). That is, it does not make sense to compare the value of $\psi(x)$ at different points. So, to write down a sensible theory including kinetic terms for ψ, we need to introduce a new derivative, D_μ, such that:

$$D_\mu \psi(x) \longrightarrow e^{i\alpha(x)} D_\mu \psi(x). \tag{10}$$

To do this, assume we have an object $U(y,x)$ that transforms under (8) as

$$U(y,x) = e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)}. \tag{11}$$

$U(y,x)$ "transports" the gauge transformation from $x \longrightarrow y$.

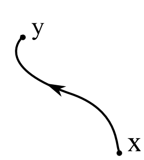

Figure 2: The parallel transport $U(y,x)$ transports the gauge transformation from x to y.

That is,

$$U(y,x)\psi(x) \longrightarrow e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)}e^{i\alpha(x)}\psi(x) = e^{i\alpha(y)}(U(y,x)\psi(x)), \tag{12}$$

transforming as $\psi(y)$. Since $\psi(y)$ and $U(y,x)\psi(x)$ have the same transformation properties, $\psi(y) - U(y,x)\psi(x)$ is well-defined.

Now take $y = x + \epsilon n$, and define

$$n^\mu D_\mu \psi = \lim_{\epsilon \to 0} \frac{\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)}{\epsilon}. \tag{13}$$

By construction,

$$D_\mu \psi \longrightarrow \lim_{\epsilon \to 0} e^{i\alpha(x+\epsilon n)} D_\mu \psi = e^{i\alpha(x)} D_\mu \psi \tag{14}$$

and

$$\mathscr{L} = -i\overline{\psi}(\gamma^\mu D_\mu - m)\psi \tag{15}$$

is invariant under (8). We now want to construct $U(y,x)$ explicitly. Since only local phase multiplication is a symmetry, $U(y,x)$ should be a phase, as we don't want to change other properties of $\psi(x)$. We begin infinitesimally:

$$U(x+\epsilon n, x) = 1 + i\epsilon n^\mu e A_\mu(x) + \ldots, \tag{16}$$

where e is a constant and $A_\mu(x)$ is a real vector field. Under the transformation (8),

$$U(x+\epsilon n, x) \longrightarrow e^{i\alpha(x+\epsilon n)} U(x+\epsilon n, x) e^{-i\alpha(x)}, \tag{17}$$

so that

$$1 + ie\epsilon n^\mu A_\mu(x) \longrightarrow e^{i\alpha(x)}(1 + i\epsilon n^\mu \partial_\mu \alpha(x))(1 + ie\epsilon n^\mu A_\mu(x))e^{-i\alpha(x)}, \tag{18}$$

and hence

$$A_\mu(x) \longrightarrow A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x). \tag{19}$$

Finally, we have for the covariant derivative D_μ:

$$D_\mu \psi = \partial_\mu \psi - ieA_\mu \psi = (\partial_\mu - ieA_\mu)\psi. \tag{20}$$

Inserting the transformation laws for $A_\mu(x)$ and $\psi(x)$: (19) and (8), respectively, we have that $D_\mu\psi(x)$ transforms as $\psi(x)$. We want $A_\mu(x)$ to be a dynamical field, and hence we require a kinetic term for this vector field, which should be invariant under (19). To construct this, we note that $D_\mu(D_\nu\psi)$ transforms as ψ, and so does $(D_\mu D_\nu - D_\nu D_\mu)\psi$, so we define

$$[D_\mu, D_\nu] = [\partial_\mu - ieA_\mu, \partial_v - ieA_\nu] \equiv -ieF_{\mu\nu}, \tag{21}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{22}$$

and we have that $F_{\mu\nu}\psi$ transforms as ψ, so that $F_{\mu\nu}$ is invariant.

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