Problem Set 7 Solutions

1. (a) Changing the variable $\bar\lambda = \bar\lambda(\lambda)$ we can find $\bar\beta$ using the chain rule of differentiation. We find,

$$\bar\beta(\bar\lambda) = \mu\frac{d\bar\lambda}{d\mu} = \mu\frac{d\lambda}{d\mu}\frac{d\bar\lambda}{d\lambda} = \beta(\lambda)\frac{d\bar\lambda}{d\lambda}\,. \tag{1}$$

Thus β transforms like a contravariant vector. For example, recall how in GR V^μ transforms like

$$V'^\mu(x') = V^\alpha(x)\,\frac{\partial x'^\mu}{\partial x^\alpha}\,. \tag{2}$$

(b) Using (1), together with

$$\bar\lambda = \lambda + a_2\lambda^2 + a_3\lambda^3 + ... \tag{3}$$

and

$$\beta = b_2\lambda^2 + b_3\lambda^3 + b_4\lambda^4 + ...\,, \tag{4}$$

we find

$$\bar\beta = (b_2\lambda^2 + b_3\lambda^3 + b_4\lambda^4..)(1 + 2a_2\lambda + 3a_3\lambda^2 + ...)\,, \tag{5}$$

which to fourth order gives

$$\bar\beta = b_2\lambda^2 + (b_3 + 2a_2b_2)\lambda^3 + (b_4 + 2a_2b_3 + 3a_3b_2)\lambda^4 + \mathcal{O}(\lambda^5)\,. \tag{6}$$

We must express $\bar\beta$ in terms of $\bar\lambda$. Instead of finding λ in terms of $\bar\lambda$ we calculate

$$\begin{aligned}\bar\lambda^2 &= \lambda^2 + 2a_2\lambda^3 + (2a_3 + a_2^2)\lambda^4 + \ldots\\ \bar\lambda^3 &= \lambda^3 + 3a_2\lambda^4 + ..\\ \bar\lambda^4 &= \lambda^4\,.\end{aligned} \tag{7}$$

and recalling that $\bar\beta = \bar b_2\bar\lambda^2 + \bar b_3\bar\lambda^3 + \bar b_4\bar\lambda^4 + \ldots$ we find that

$$\bar{b_2} = b_2, \quad \bar{b_3} = b_3\,, \quad \text{and} \quad \bar{b_4} = b_4 - a_2b_3 + a_3b_2 - a_2^2b_2\,. \tag{8}$$

so that we can make $\bar b_4$ anything we want by tweaking the as.

The fixed point of a renormalization flow is defined by $\beta(\lambda_f) = 0$ which implies $\bar\beta(\bar\lambda_f) = 0$ (if $\frac{d\bar\lambda}{d\lambda}$ and $\frac{d\lambda}{d\bar\lambda}$ exist, which we assume). Finally

$$\bar\beta'(\bar\lambda) = \frac{d}{d\bar\lambda}\Big(\beta(\lambda)\frac{d\bar\lambda}{d\lambda}\Big) = \frac{d}{d\lambda}\Big(\beta(\lambda)\frac{d\bar\lambda}{d\lambda}\Big)\frac{d\lambda}{d\bar\lambda} = \beta'(\lambda) + \beta(\lambda)\frac{d\lambda}{d\bar\lambda}\frac{d^2\bar\lambda}{d\lambda^2}\,. \tag{9}$$

so that, at a fixed point

$$\bar\beta'(\bar\lambda_f) = \beta'(\lambda_f)\,. \tag{10}$$

(c) We have

$$\mu\frac{dg}{d\mu} = -bg^2 - cg^3 - dg^4 + \ldots\,. \tag{11}$$

For convenience, redefining the coupling and the scale

$$\lambda \equiv bg\,, \quad \text{and} \quad t \equiv \log(\mu/\mu_1)\,, \tag{12}$$

we rewrite the differential equation as

$$\frac{d\lambda}{dt} = -\lambda^2 - \frac{c}{b^2}\lambda^3 - \frac{d}{b^3}\lambda^4 - \dots . \tag{13}$$

We then rearrange as

$$dt = -\frac{d\lambda}{\lambda^2}\frac{1}{\left(1+\frac{c}{b^2}\lambda+\frac{d}{b^3}\lambda^2+\dots\right)} = -\frac{d\lambda}{\lambda^2} + \frac{c}{b^2}\frac{d\lambda}{\lambda} + \dots \tag{14}$$

This equation can be integrated to yield

$$t = \text{constant} + \frac{1}{\lambda} + \frac{c}{b^2}\ln\lambda + \mathcal{O}(\lambda) + .. , \tag{15}$$

where the constant is independent of t (i.e. μ). This means that the constant is RG invariant. Rewriting the result in the original variables, and absorbing $\ln\mu_1$ into the definition of the constant, we have

$$\ln\mu = \text{constant} + \frac{1}{bg(\mu)} + \frac{c}{b^2}\log[bg(\mu)] + O(g(\mu)) . \tag{16}$$

Setting the constant equal to the RG invariant scale Λ, we get

$$\ln\frac{\mu}{\Lambda} = \frac{1}{bg(\mu)} + \frac{c}{b^2}\log[bg(\mu)] + O(g(\mu)) . \tag{17}$$

(d) In a generic QFT one would have $\beta = \beta\left(g(\mu), \frac{m_1}{\mu}, \dots\right)$ where m_1 is a parameter of the theory with $[m_1] = 1$. Contrary to the most general form, in a theory without mass parameters, we have

$$\beta(g) = \mu\frac{dg}{d\mu} \tag{18}$$

which implies

$$\int\frac{dg}{\beta(g)} = \int\frac{d\mu}{\mu} \tag{19}$$

integrating both sides give some function of g on the left (which contains no dimensionful parameters), and a log with a universal constant scale of integration on the right giving

$$F(g) = \log\left(\frac{\mu}{\Lambda}\right) \tag{20}$$

Inverting this relation gives that g is some function of the log, i.e.

$$g = f\left(\log\left(\frac{\mu}{\Lambda}\right)\right) . \tag{21}$$

2. The Lagrangian is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}(\partial\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 + \frac{g_0}{6}\phi_0^6 \\ &= -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{g\mu^{\frac{\epsilon}{2}}}{6}\phi^6 - \frac{A}{2}(\partial\phi)^2 - \frac{B}{2}m^2\phi^2 + \frac{g\mu^{\frac{\epsilon}{2}}C}{6}\phi^6\end{aligned} \tag{22}$$

with

$$g_0 = g\mu^{\frac{\epsilon}{2}}(1+A)^{-\frac{3}{2}}(1+C) \tag{23}$$

Thus to calculate the beta function, we need only calculate A and C. This involves calculating the momentum dependent part of the two point function, and the three point function. Both of these relevant computations have been done in class and/or in previous problem sets. In class, we found that the two point function was (with $D = m^2 + x(1-x)p^2$)

$$\begin{aligned}
&= \frac{g^2\mu^\epsilon\Gamma(2-\frac{d}{2})}{2(4\pi)^{\frac{d}{2}}}\int_0^1 dx\,\frac{1}{D^{2-\frac{d}{2}}} - Ap^2 - m^2B \\
&= \text{finite} - \frac{\alpha}{\epsilon}\int_0^1 D\,dx - Ap^2 - Bm^2 \\
&= \text{finite or p independent} - \frac{\alpha p^2}{6\epsilon} - Ap^2 \qquad (24)
\end{aligned}$$

thus $A = -\frac{\alpha}{6\epsilon}$ in a minimal subtraction scheme. Similarly, in the previous homework we found that the three point function was

$$\begin{aligned}
&= g + gC + \text{finite} + g\frac{2\alpha}{\epsilon}\int dx\,dy\,dz\,\delta(x+y+z-1) \\
&= \text{finite} + g\left(C + \frac{\alpha}{\epsilon}\right) \qquad (25)
\end{aligned}$$

thus in MS, $C = -\frac{\alpha}{\epsilon}$. To lowest non trivial order

$$\begin{aligned}
g_0 &= g\mu^{\frac{\epsilon}{2}}\left(1-\frac{\alpha}{6\epsilon}\right)^{-\frac{3}{2}}\left(1-\frac{\alpha}{\epsilon}\right) \\
&= g\mu^{\frac{\epsilon}{2}}\left(1-\frac{3\alpha}{4\epsilon}\right) \qquad (26)
\end{aligned}$$

Using the scale independence of bare quantities, we log differentiate each side, which gives

$$0 = \beta_g\mu^{\frac{\epsilon}{2}}\left(1-\frac{9\alpha}{4\epsilon}\right) + \frac{g\epsilon}{2}\left(1-\frac{3\alpha}{4\epsilon}\right) \qquad (27)$$

Multiplying by $(1+\frac{9\alpha}{4\epsilon})$ and taking the $\epsilon \to 0$ limit gives

$$\beta_g = -\frac{3}{4}\alpha g \qquad (28)$$

Using the chain rule and the definition of α gives

$$\beta_\alpha = -\frac{3}{2}\alpha^2 + O(\alpha^3) \qquad (29)$$

which implies that the theory is asymptotically free.

3. (a) We have

$$\mathcal{L} = \frac{1}{2}\left[(\partial\phi_1)^2 + (\partial\phi_2)^2\right] - \frac{\lambda}{4!}(\phi_1^4+\phi_2^4) - \frac{2\rho}{4!}(\phi_1^2\phi_2^2) \qquad (30)$$

We can use most of the results we have derived previously in the course with regard to ϕ^4 theory,the only subtlety here is with symmetry factors. Once again each λ vertex comes with a factor of $-i\lambda$. However, because there are fewer combinatorial symmetries, each ρ vertex comes with a factor of $-i4\frac{2\rho}{4!} = -i\frac{\rho}{3}$. Let us renormalize the λ vertex. For concreteness, consider ϕ_1 external lines (it does not matter which). We will have s, t, u channel contributions with both types of particles running in the loop. The result will give something like PS (10.21)

$$= -i\lambda - i(\lambda^2 + \frac{\rho^2}{9})(V(t)+V(s)+V(u)) - i\delta_\lambda \qquad (31)$$

All Vs have the same infinite part in dimensional regularization (see eg PS (10.23))

$$V \sim -\frac{1}{16\pi^2\epsilon} + \text{finite} \qquad (32)$$

which implies, in the MS scheme

$$\delta_\lambda = \left(\lambda^2 + \frac{\rho^2}{9}\right)\frac{3}{16\pi^2\epsilon} \tag{33}$$

We thus have (there is no wavefunction renormalization coming from the two point function at one-loop order)

$$\begin{aligned}\lambda_0 &= \mu^\epsilon(\lambda + \delta_\lambda)\\ &= \mu^\epsilon\left(\lambda + \left(\lambda^2 + \frac{\rho^2}{9}\right)\frac{3}{16\pi^2\epsilon}\right)\end{aligned} \tag{34}$$

which implies

$$0 = \beta_\lambda(1+2\lambda) + \epsilon\left(\lambda + \left(\lambda^2 + \frac{\rho^2}{9}\right)\frac{3}{16\pi^2\epsilon}\right) + \frac{2}{9}\rho\beta_\rho\frac{3}{16\pi^2\epsilon} \tag{35}$$

Rearranging, and using the fact that the beta functions are $O(\lambda,\rho)$, to lowest order this implies

$$\beta_\lambda = -\epsilon\left(\lambda + \left(-\lambda^2 + \frac{\rho^2}{9}\right)\frac{3}{16\pi^2\epsilon}\right) - \frac{2}{9}\rho\beta_\rho\frac{3}{16\pi^2\epsilon} \tag{36}$$

We now calculate the renormalization of ρ. Two of the loop diagrams are $O(\lambda\rho)$ and have one type of particle running in the loop. The other two are $O(\rho^2)$ and have two types of particles in the loop. We must multiply these last two diagrams by a factor of two to account for the change in symmetry factor. In total we get

$$\begin{aligned}&= \text{finite} - i\frac{\lambda\rho}{3}2V - i\frac{\rho^2}{9}4V - i\frac{\delta_\rho}{3}\\ &= \text{finite} - \frac{2}{3}i\lambda\rho\left(-\frac{1}{16\pi^2\epsilon}\right) - i\rho^2\frac{4}{9}\left(-\frac{1}{16\pi^2\epsilon}\right) - i\frac{\delta_\rho}{3}\end{aligned} \tag{37}$$

which implies, in the MS scheme

$$\begin{aligned}\delta_\rho &= \frac{\lambda\rho}{8\pi^2\epsilon} + \frac{\rho^2}{12\pi^2\epsilon}\\ \rho_0 &= \rho\mu^\epsilon\left(1 + \frac{\rho}{8\pi^2\epsilon}\left(\lambda + \frac{2}{3}\rho\right)\right)\end{aligned} \tag{38}$$

Once again, differentiating and manipulating to lowest order, we get

$$\beta_\rho = -\epsilon\rho\left(1 + \frac{1}{8\pi^2\epsilon}\left(-\frac{2}{3}\rho\right)\right) - \rho\beta_\lambda\frac{1}{8\pi^2\epsilon} \tag{39}$$

Substituting (39) into (36) to lowest order and taking the $\epsilon \to 0$ limit gives

$$\beta_\lambda = \frac{3}{16\pi^2}\left(\lambda^2 + \frac{\rho^2}{9}\right) \tag{40}$$

substituting in the opposite way gives

$$\beta_\rho = \frac{1}{8\pi^2}\left(\lambda\rho + \frac{2}{3}\rho^2\right) \tag{41}$$

(b) Using the chain rule, we get

$$\begin{aligned}\beta_{\frac{\rho}{\lambda}} &= \frac{1}{\lambda^2}(\beta_\rho\lambda - \beta_\lambda\rho)\\ &= -\frac{\lambda}{48\pi^2}\frac{\rho}{\lambda}\left(\frac{\rho}{\lambda} - 3\right)\left(\frac{\rho}{\lambda} - 1\right)\end{aligned} \tag{42}$$

which has zeros at $\rho/\lambda = \{0, 1, 3\}$. We wrote the result in this form, because stability implies that $\lambda > 0$, while it is harder to say anything definite about the sign of ρ. This beta function is positive for $\rho/\lambda < 0$ and for $1 < \rho/\lambda < 3$, and negative elsewhere which implies that $\rho/\lambda = 1$ is an IR stable fixed point, while the others are IR unstable. (For negative $\rho/\lambda < 0$ we flow to strong coupling, where perturbation theory isn't controlled anymore.) Thus, if we start with the $0 < \rho/\lambda < 3$ it will flow to this fixed point, while $\rho/\lambda > 3$, the coupling will flow away to infinity.

(c) We have already established the existence of these fixed points; only the asymptotically symmetric one is IR stable. In $4-\epsilon$ dimensions, we can read off the beta functions from above (before taking the $\epsilon \to 0$ limit)

$$\beta_\lambda = -\epsilon\lambda + \left(\lambda^2 + \frac{\rho^2}{9}\right)\frac{3}{16\pi^2} \tag{43}$$

$$\beta_\rho = -\epsilon\rho + \frac{1}{8\pi^2}\left(\lambda\rho + \frac{2}{3}\rho^2\right) \tag{44}$$

Your sketch should involve a simultaneous plot of λ and ρ, or a line of ρ/λ with some arrows to indicate the flow to the IR. At finite but small ϵ, you should find that there is a stable IR fixed point at finite ρ, λ. In my plot the fixed points are (obtained from the zeros of the β functions):

$$\lambda_A = 0 \qquad \rho_A = 0 \qquad A \text{ is IR unstable} \tag{45}$$

$$\lambda_B = \frac{16\pi^2}{3}\epsilon \qquad \rho_B = 0 \qquad B \text{ has one unstable direction} \tag{46}$$

$$\lambda_C = \frac{8\pi^2}{3}\epsilon \qquad \rho_C = 8\pi^2\epsilon \qquad C \text{ has one unstable direction} \tag{47}$$

$$\lambda_D = \frac{24\pi^2}{5}\epsilon \qquad \rho_D = \frac{24\pi^2}{5}\epsilon \qquad D \text{ is IR stable.} \tag{48}$$

We conclude that we get an emergent $O(2)$ symmetry asymptotically in the IR, if we start from the appropriate parameter range. The basin of attraction is called universality class in the statistical physics literature. By tuning we can get a large range of energy, where the fixpoints B or C determine the behavior of the theory. There are other universality classes in this space of couplings that we cannot explore with the ϵ expansion techniques, as they lie in the strong coupling regime of the theory.

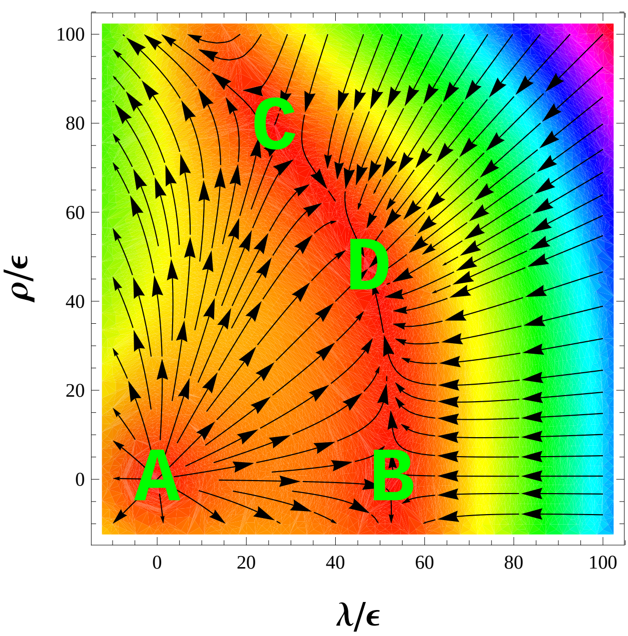

FIG. 1. The RG flow of the theory.

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Fall 2010