Problem Set 6 Solutions

1. (a) Under charge conjugation, as expected, the electromagnetic current changes sign, $Cj^\mu C^\dagger = -j^\mu$. QED is C invariant, implying both that the Lagrangian is invariant under C, hence $Cj^\mu A_\mu C^\dagger = j^\mu A_\mu$ and $CA^\mu C^\dagger = -A^\mu$, and that the vacuum is invariant under C, $C|0\rangle = |0\rangle$. Thus, the vacuum expectation value of any odd number of currents and fields is (inserting many copies of $C^\dagger C = 1$),

$$\begin{aligned}\langle 0|T\left\{A^{\mu_1}\dots A^{\mu_{k-1}}j^{\mu_k}\dots j^{\mu_{2n+1}}\right\}|0\rangle &= \langle 0|T\left\{C^\dagger C A^{\mu_1}C^\dagger\dots Cj^{\mu_{2n+1}}C^\dagger C\right\}|0\rangle \\ &= (-1)^{2n+1}\langle 0|T\left\{A^{\mu_1}\dots j^{\mu_{2n+1}}\right\}|0\rangle \\ &= -\langle 0|T\left\{A^{\mu_1}\dots j^{\mu_{2n+1}}\right\}|0\rangle \qquad (1)\end{aligned}$$

which implies this expression is zero.

(b) The one-loop diagram contributing to the one-point function is shown in Figure 1. The amplitude is:

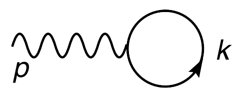

FIG. 1. tadpole diagram in QED.

$$\langle 0|A^\mu(p=0)|0\rangle = (-1)\int \frac{d^dk}{(2\pi)^d}\frac{\mathrm{tr}\left[(-ie\gamma^\mu)i(\not{k}+m)\right]}{k^2-m^2+i\epsilon} = (-e)\int \frac{d^dk}{(2\pi)^d}\frac{d\,k^\mu}{k^2-m^2+i\epsilon}. \qquad (2)$$

Symmetric integration then gives us zero.

There are two Feynman diagrams contributing to the three-point function at one-loop order as shown in Figure 2.

The first one is equal to:

$$e^3\int \frac{d^dk}{(2\pi)^d}\frac{\mathrm{tr}\left[(\not{q}_1+m)\gamma^{\mu_1}(\not{q}_2+m)\gamma^{\mu_2}(\not{q}_3+m)\gamma^{\mu_3}\right]}{(q_1^2-m^2+i\epsilon)(q_2^2-m^2+i\epsilon)(q_3^2-m^2+i\epsilon)}, \qquad (3)$$

where

$$\begin{aligned}q_1 &= k,\\ q_2 &= k-p_1,\\ q_3 &= k-p_1+p_2. \qquad (4)\end{aligned}$$

The second diagram is obtained by switching vertices 2 and 3, or equivalently by reversing the the arrows in the fermion loop. Also note the opposite flow of the momentum.

$$e^3\int \frac{d^dk}{(2\pi)^d}\frac{\mathrm{tr}\left[(-\not{q}_2+m)\gamma^{\mu_1}(-\not{q}_1+m)\gamma^{\mu_3}(-\not{q}_3+m)\gamma^{\mu_2}\right]}{(q_1^2-m^2+i\epsilon)(q_2^2-m^2+i\epsilon)(q_3^2-m^2+i\epsilon)}, \qquad (5)$$

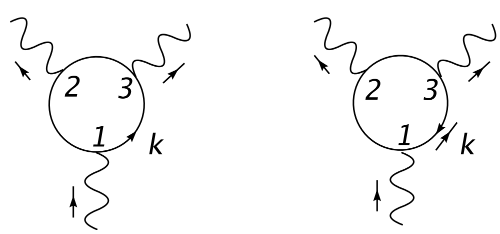

FIG. 2. Diagrams contributing to the three-point function at one-loop in QED.

Using the identity (A.28) from PS and the cyclic property of the trace we get:

$$\mathrm{tr}\left[(-\not{q}_2+m)\gamma^{\mu_1}(-\not{q}_1+m)\gamma^{\mu_3}(-\not{q}_3+m)\gamma^{\mu_2}\right]=\mathrm{tr}\left[(-\not{q}_1+m)\gamma^{\mu_1}(-\not{q}_2+m)\gamma^{\mu_2}(-\not{q}_3+m)\gamma^{\mu_3}\right] \tag{6}$$

We observe that we get the contribution of the second diagram by the replacement $q_i \to -q_i$. The non-zero contributions to the trace come from an even number of gamma matrices. Since the number of uncontracted gamma matrices is odd (because of an odd number of external photons), we need an odd number of q's for a non-zero trace. Now, the second diagram is obtained by $q_i \to -q_i$ and as a result, the sum of the two diagrams is zero.

2. (a) In momentum space, the quadratic part of the Lagrangian containing the vector field is:

$$\mathcal{L}_{2,A}=-\frac{1}{2}A_\mu[-(k^2-m^2)\eta^{\mu\nu}+k^\mu k^\nu]A_\nu \tag{7}$$

Inverting this quadratic form we get

$$D_{\nu\sigma}(k)=\frac{-i}{k^2-m^2+i\epsilon}\left[\eta_{\nu\sigma}-\frac{k_\nu k_\sigma}{m^2}\right] \tag{8}$$

(b) We can take the large momentum limit by rescaling $k \to \lambda k$ to get

$$D_{\nu\sigma}(\lambda k)\xrightarrow[(\lambda\to\infty)]{}\frac{ik_\nu k_\sigma}{m^2k^2} \tag{9}$$

Indeed (8) is independent of λ for large λ and as far as power counting goes $D_{\nu\sigma}(k)$ scales as $\mathcal{O}(1)$.

(c) Because of the $O(1)$ scaling of the photon propagator, the superficial degree of divergence is now given by

$$D=dL-I_e \tag{10}$$

with d the number of spacetime dimensions, L the number of loops, and I_e the number of internal photon lines. We can use $L=I_e+I_A-V+1$ and $V=2I_A+E_A=I_e+\frac{1}{2}E_e$, to eliminate the counting of the internal propagators. The result is

$$D=d+\frac{d-2}{2}V-\frac{d-1}{2}E_e-\frac{d}{2}E_A \tag{11}$$

(d) For four dimensions, the superficial degree of divergence is

$$D=4+V-\frac{3}{2}E_e-2E_A \tag{12}$$

The formula for superficial degree of divergence given in class is

$$D=4-\Delta_eE_e-\Delta_AE_A-[e]V \tag{13}$$

It is still the case that $\Delta_e=[\psi]=3/2$, but now $\Delta_A=[A]=2$. The latter implies that $[e]=-1$, and so the two equations agree.

(e) The theory is **naively** non-renomalizable as all amplitudes diverge at a high enough order in perturbation theory as can be seen from (12). This implies we need an infinite number of counterterms. I gave full credit for this answer, however, the theory turns out to renormalizable. The global $U(1)$ symmetry and the Ward identities derived from it ensure that the problematic part of the vector propagator doesn't play a role in loops. A_μ couples to a conserved $U(1)$ current j_μ and hence the conservation of current implies that the $k_\nu k_\rho$ part of the propagator is removed and the QED power counting is restored. What happens is that the superficial degree of divergence is a bad guide for this theory, symmetry forbids many expected divergent terms. Thus, insofar renormalizability is conserved the theory is equivalent to QED and is renormalizable. A comprehensive reference to the proof of renormalizability is *hep-th/0305076*.

3. (a) The operator product expansion is understood to be valid inside (time-ordered) correlation functions, hence in the free theory case one can apply Wick's theorem in these calculations. The direct application of Wick's theorem gives us

$$\begin{aligned}\phi^4(x) = \lim_{\epsilon\to 0}\phi(x+3\epsilon)\phi(x+\epsilon)\phi(x-\epsilon)\phi(x-3\epsilon) =: \phi^4(x) : + [3D(2\epsilon) + 2D(4\epsilon) + D(6\epsilon)] : \phi^2(x) : \\ + \left[D(2\epsilon)^2 + D(2\epsilon)D(6\epsilon) + D(4\epsilon)^2\right]\mathbb{I}\end{aligned} \tag{14}$$

where we took all possible contractions and could safely take the $\epsilon \to 0$ limit inside the normal-ordering symbol. Writing down the most singular pieces we get:

$$D(\epsilon) = \frac{1}{4\pi^2\epsilon^2} + \dots \tag{15}$$

$$\phi^4(x) = : \phi^4(x) : + \frac{65}{288\pi^2\epsilon^2} : \phi^2(x) : + \frac{169}{36864\pi^4\epsilon^4}\mathbb{I} \, . \tag{16}$$

(b) Wick's theorem for the product of normal ordered strings of operators states that we only have to take all possible contractions *between* the two strings. The pairings within the normal-ordered string itself was taken care of by the normal ordering. [1] This algorithm gives us the following $\Delta \leq 4$ terms in the OPE:

$$\begin{aligned}: \phi^4(x) :: \phi^4(0) : &= 4! D(x)^4 \, \mathbb{I} + \frac{(4!)^2}{3!} D(x)^3 : \phi(x)\phi(0) : + \frac{(4\cdot 3)^2}{2} D(x)^2 : \phi^2(x)\phi^2(0) : + \dots \\ &= \sum_{k=1}^{4} \frac{(4!)^2}{(4-k)!(k!)^2} D(x)^{4-k} : \phi^k(x)\phi^k(0) :\end{aligned} \tag{17}$$

These kind of results can be elegantly derived from the following identity by the generating function method (Di Francesco: CFT (6.63)):

$$: e^{\alpha\phi(x)} :: e^{\beta\phi(0)} := e^{\alpha\beta D(x)} : e^{\alpha\phi(x)+\beta\phi(0)} : \tag{18}$$

This identity plays a rather important role in the study of vertex operators in the field of CFT and String Theory.

Now what remains to be done is to replace : $\phi^k(x)\phi^k(0)$: with composite operators multiplied by x-dependent coefficients. This can be done by Taylor expanding for small xs. Keeping only $\Delta \leq 4$ operators we get:

$$\begin{aligned}: \phi^4(x) :: \phi^4(0) :=& 4! D(x)^4 \, \mathbb{I} + \frac{(4!)^2}{3!} D(x)^3 \left[: \phi^2(0) : + x^\mu : \partial_\mu\phi(0)\phi(0) : + \frac{1}{2} x^\mu x^\nu : \partial_\mu\partial_\nu\phi(0)\phi(0) : + \dots\right] \\ &+ \frac{(4\cdot 3)^2}{2} D(x)^2 : \phi^4(0) : + \dots\end{aligned} \tag{19}$$

[1] A little bit of digression is in order here: in perturbation theory on the level of the diagrammatic calculation normal ordering means that our regularization subtracted loops that only have one internal propagator In a two dimensional scalar theory this can be shown to sufficient to take care of all the divergences and thus is a valid renormalization procedure.

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Fall 2010