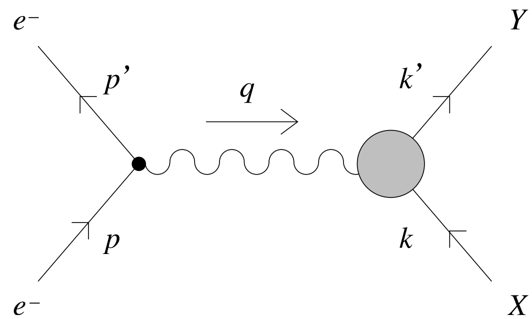
Problem Set 5 Solutions

1. We will use PS conventions in this Problem Set. We consider the scattering of high energy electrons from a target, a process which can be described to leading order in α by the following generic diagram:



The gray filled circle represents the coupling of the virtual photon to the target, taking the target from an initial state X to some final state Y, which we incorporate into a scattering amplitude factor $\hat{\mathcal{M}}^\nu(q)$, where $q = p - p' = k' - k$. The full matrix element for the process is:

$$\mathcal{M} = (-ie)\bar{u}(p')\gamma^\mu u(p)\frac{-ig_{\mu\nu}}{q^2}\hat{\mathcal{M}}^\nu(q) \tag{1}$$

Parts a) and b) of the problem analyze the structure of the $\bar{u}(p')\gamma^\mu u(p)$ piece, the results of which are used in c), d), and e) to find an expression for the cross-section.

(a)

$$\bar{u}(p')\gamma^\mu u(p) = Aq^\mu + B\tilde{q}^\mu + C\epsilon_1^\mu + D\epsilon_2^\mu \tag{2}$$

Dotting with q_μ:

$$\begin{aligned} \text{LHS} &= \bar{u}(p')\not{q}u(p) = \bar{u}(p')(\not{p}' - \not{p})u(p) = (m-m)\bar{u}(p')u(p) = 0 \\ \text{RHS} &= Aq^2 + Bq\cdot\tilde{q} \end{aligned} \tag{3}$$

where the last line follows from the fact that $q \cdot \epsilon_i = 0$. Thus,

$$\frac{B}{A} = -\frac{q^2}{q\cdot\tilde{q}}\,. \tag{4}$$

In the ultra relativistic limit for the electrons the kinematics for this problem are as follows:

$$\begin{aligned} p &= (E, E\hat{z}) \\ p' &= (E', E'\sin\theta, 0, E'\cos\theta) \\ q &= p - p' = (E - E', -E'\sin\theta, 0, E - E'\cos\theta) \\ \tilde{q} &= (q^0, -\mathbf{q}) = (E - E', E'\sin\theta, 0, E'\cos\theta - E)\,. \end{aligned} \tag{5}$$

In the small θ limit, we can calculate

$$\begin{aligned} q^2 &= -2EE'(1-\cos\theta) \approx -EE'\theta^2 \\ q\cdot\tilde{q} &= (E-E')^2 + E'^2\sin^2\theta + (E - E'\cos\theta)^2 \approx 2(E-E')^2\,. \end{aligned} \tag{6}$$

We see that $\frac{B}{A} = -\frac{q^2}{q\cdot\tilde{q}}$ will be at most $\mathcal{O}\left(\theta^2\right)$, and so B can be neglected in the rest of the analysis. As noted in the problem A is irrelevant because of the Ward identity $q_\mu\hat{\mathcal{M}}^\mu(q) = 0$, which is just the consequence of the gauge invariance of QED. We conclude that it is enough to determine C and D to solve the forward scattering problem.

(b) We want to compute explicitly $\bar{u}(p')\gamma\cdot\epsilon_i u(p)$ for massless electrons, where $u(p)$ and $u(p')$ are spinors of definite helicity. I will abbreviate $u(p)$ as u and $\bar{u}(p')$ as $\bar{u}'$ for rest of the problem. First, we will use method of spinor products developed in Problem 5.3 of PS to evaluate $\bar{u}'\gamma^i u$ for both left and right helicity. For this we need to choose a light-like vector K which is independent of p and p'. An obvious choice for K is $(1,0,1,0)$.

$$\begin{aligned}\bar{u}'_L\gamma^i u_L &= \frac{1}{\sqrt{(2p'\cdot K)(2p\cdot K)}}\bar{u}_{R0}\not{p}'\gamma^i\not{p}u_{R0}\\ &= \frac{1}{\sqrt{(2p'\cdot K)(2p\cdot K)}}Tr(P_R\not{k}_0\not{p}'\gamma^i\not{p})\\ &= \frac{1}{\sqrt{EE'}}[(K\cdot p')p^i-(p\cdot p')K^i+(K\cdot p)p'^i-i\epsilon^{\mu\nu i\rho}K_\mu p'_\nu p_\rho].\end{aligned} \tag{7}$$

Similarly,

$$\bar{u}'_R\gamma^i u_R = \frac{1}{\sqrt{EE'}}[(K\cdot p')p^i-(p\cdot p')K^i+(K\cdot p)p'^i+i\epsilon^{\mu\nu i\rho}K_\mu p'_\nu p_\rho]. \tag{8}$$

Notice that only ϵ-tensor term changes sign. Other helicity combinations $\bar{u}'_L\gamma^i u_R$ and $\bar{u}'_R\gamma^i u_L$ vanish either by simple calculation or by the conservation of helicity for massless fermions. Applying kinematics to above expressions we get

$$\begin{array}{lll}\bar{u}'_L\gamma^1 u_L = \sqrt{EE'}\ \theta & \bar{u}'_L\gamma^2 u_L = +i\sqrt{EE'}\ \theta & \bar{u}'_L\gamma^3 u_L = \sqrt{EE'}(2-i\theta)\\ \bar{u}'_R\gamma^1 u_R = \sqrt{EE'}\ \theta & \bar{u}'_R\gamma^2 u_R = -i\sqrt{EE'}\ \theta & \bar{u}'_R\gamma^3 u_R = \sqrt{EE'}(2+i\theta).\end{array} \tag{9}$$

We can work out what ϵ_1 and ϵ_2 need to be by requiring they be perpendicular to q and $\tilde{q}$. Requiring that ϵ_1 be perpendicular to the plane of scattering and ϵ_2 to be in the plane of scattering gives

$$\epsilon_1^\mu = \begin{pmatrix}0\\1\\0\\\frac{E'\theta}{E-E'}\end{pmatrix} \qquad \epsilon_2^\mu = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \tag{10}$$

Therefore

$$\bar{u}'_L\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon_1}u_L = \bar{u}'_R\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon_1}u_R = \frac{E+E'}{E-E'}\sqrt{EE'}\ \theta \tag{11}$$

$$\bar{u}'_L\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon_2}u_L = -\bar{u}'_R\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon_2}u_R = i\sqrt{EE'}\ \theta. \tag{12}$$

Secondly, we will use a more direct approach to get the same result. With the basis choice of PS for the Dirac matrices the definite helicity states of the electrons and photon are:

$$\gamma^\mu = \begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix} \tag{13}$$

$$u_L = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad u_R = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \tag{14}$$

$$u'_L = \sqrt{2E'}\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\\0\\0\end{pmatrix} \approx \sqrt{2E'}\begin{pmatrix}-\frac{\theta}{2}\\1\\0\\0\end{pmatrix} \qquad u'_R = \sqrt{2E'}\begin{pmatrix}0\\0\\1\\\frac{\theta}{2}\end{pmatrix} \tag{15}$$

$$\epsilon_L^\mu = \frac{1}{\sqrt{2}}(\epsilon_1^\mu - i\epsilon_2^\mu) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\\frac{E'\theta}{E-E'}\end{pmatrix} \qquad \epsilon_R^\mu = \frac{1}{\sqrt{2}}(\epsilon_1^\mu + i\epsilon_2^\mu) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\\frac{E'\theta}{E-E'}\end{pmatrix} \tag{16}$$

Not forgetting that the photon is outgoing (in the subprocess) we get

$$\bar{u}'_L\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon^*_L}u_L = \bar{u}'_R\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon^*_R}u_R = \frac{\sqrt{2}E'}{E-E'}\sqrt{EE'}\ \theta \tag{17}$$

$$\bar{u}'_L\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon^*_R}u_L = \bar{u}'_R\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon^*_L}u_R = \frac{\sqrt{2}E}{E-E'}\sqrt{EE'}\ \theta\ . \tag{18}$$

This result makes the parity invariance of the theory manifest (i.e. flipping all the helicities doesn't change the result) and can be easly checked to agree with the result of the spinor product method.

(c) The expansion of the spinor product

$$\bar{u}(p')\gamma^\mu u(p) = Aq^\mu + B\tilde{q}^\mu + C\epsilon_1^\mu + D\epsilon_2^\mu \tag{19}$$

works because $\bar{u}(p')\gamma^\mu u(p)$ is a 4-vector, and q^μ, $\tilde{q}^\mu$, ϵ_1^μ, and ϵ_2^μ consitute a basis of four linearly independent 4-vectors. The coefficients are just the projection of $\bar{u}(p')\gamma^\mu u(p)$ onto each of the basis vectors. From the argument in part (a) $B = \mathcal{O}(\theta^2)$ and in the small θ limit can be neglected. Hence our scattering matrix element $\mathcal{M}$ can be written:

$$\begin{aligned}\mathcal{M} &= -\frac{e}{q^2}\bar{u}(p')\gamma^\mu u(p)\hat{\mathcal{M}}_\mu(q) \\ &= -\frac{e}{q^2}(C\epsilon_1^\mu + D\epsilon_2^\mu)\hat{\mathcal{M}}_\mu(q)\,.\end{aligned} \tag{20}$$

Here the A term drops out because $q^\mu\hat{\mathcal{M}}_\mu = 0$ by the Ward identity. The coefficients C and D are just the projections

$$\begin{aligned}C &= \bar{u}'\boldsymbol{\gamma}.\boldsymbol{\epsilon_1}\ u \\ D &= \bar{u}'\boldsymbol{\gamma}.\boldsymbol{\epsilon_2}\ u\end{aligned} \tag{21}$$

so we can write:

$$\begin{aligned}\mathcal{M} &= -\frac{e}{q^2}\left[\bar{u}(p')\boldsymbol{\gamma}.\boldsymbol{\epsilon_1}u(p)\epsilon_1^\mu\hat{\mathcal{M}}_\mu(q) + \bar{u}(p')\boldsymbol{\gamma}.\boldsymbol{\epsilon_2}u(p)\epsilon_2^\mu\hat{\mathcal{M}}_\mu(q)\right] \\ &= -\frac{e}{q^2}\left[\bar{u}(p')\boldsymbol{\gamma}.\boldsymbol{\epsilon_1}u(p)\hat{\mathcal{M}}^1 + \bar{u}(p')\boldsymbol{\gamma}.\boldsymbol{\epsilon_2}u(p)\hat{\mathcal{M}}^2\right]\end{aligned} \tag{22}$$

where we introduce the notation $\hat{\mathcal{M}}_1 \equiv \epsilon_1^\mu\hat{\mathcal{M}}_\mu(q)$ and $\hat{\mathcal{M}}_2 \equiv \epsilon_2^\mu\hat{\mathcal{M}}_\mu(q)$. Using the results from part (b), we can evaluate $\mathcal{M}$ for the various helicity cases:

$$\begin{aligned}\mathcal{M}(e_R^- \to e_R^-) &= \frac{e}{q^2}\left(\frac{E+E'}{E-E'}\sqrt{EE'}\theta\hat{\mathcal{M}}_1 + i\theta\sqrt{EE'}\hat{\mathcal{M}}_2\right) \\ \mathcal{M}(e_L^- \to e_L^-) &= \frac{e}{q^2}\left(\frac{E+E'}{E-E'}\sqrt{EE'}\theta\hat{\mathcal{M}}^1 - i\theta\sqrt{EE'}\hat{\mathcal{M}}^2\right) \\ \mathcal{M}(e_L^- \to e_R^-) &= \mathcal{M}(e_R^- \to e_L^-) = 0\,.\end{aligned} \tag{23}$$

The spin-averaged (or helicity-averaged) squared matrix element is given by

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^2}{q^4}EE'\theta^2\left(\left(\frac{E+E'}{E-E'}\right)^2|\hat{\mathcal{M}}_1|^2 + |\hat{\mathcal{M}}_2|^2\right)\,. \tag{24}$$

(Note that this result sums over the outgoing photon polarizations, as $\mathcal{M}$s themselves were sums over photon polarizations.) The total scattering cross-section in the CM frame, using equation (4.79) from PS, is:

$$\sigma = \frac{1}{2E\,2E_X(1+v_X)}\int d\Pi_Y\int\frac{d^3p'}{(2\pi)^3}\frac{1}{2E'}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 \tag{25}$$

where E_X and v_X are the initial energy and velocity of the target, and $\int d\Pi_Y = \int\frac{d^3k'}{(2\pi)^3}\frac{1}{2E_Y'}(2\pi)^4\delta^4(p+k-p'-k)$ is a phase space integral over the final state Y of the target. For simplicity, let us introduce the notation

$$\int d\Pi_Y' \equiv \frac{1}{2E_X(1+v_X)}\int d\Pi_Y\,. \tag{26}$$

Rewriting the integral over p', the cross-section becomes:

$$\sigma = \frac{1}{2E}\int d\Pi_Y'\int\frac{2\pi p_\perp' dp_\perp' dp_z'}{(2\pi)^3}\frac{1}{2E'}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 \tag{27}$$

where $p'_z = E'\cos\theta$ and $p'_\perp = E'\sin\theta$. The cross section is dominated by forward scattering, it has a collinear singularity that will be regulated in part (d). Hence the dominant contribution comes from the domain, where θ is between 0 and some small θ_{max}. In this limit $p'_z \approx E'$ and $p'_\perp \approx E'\theta$, so $dp'_z \approx dE'$ and $dp'_\perp \approx E'd\theta$. Our integral for σ can be expressed as:

$$\begin{aligned}
\sigma &\approx \frac{1}{4E}\int d\Pi'_Y \int \frac{dE'}{(2\pi)^2}\int_0^{\theta_{\text{max}}} d\theta\, E'\theta\, \frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 \\
&= \frac{e^2}{4E^2}\int d\Pi'_Y \int \frac{dE'}{(2\pi)^2}\int_0^{\theta_{\text{max}}} d\theta\, \frac{E^2 E'^2\theta^3}{q^4}\left(|\hat{\mathcal{M}}^2|^2 + \left(\frac{E+E'}{E-E'}\right)^2 |\hat{\mathcal{M}}^1|^2\right) \\
&= \frac{e^2}{4E^2}\int d\Pi'_Y \int \frac{dE'}{(2\pi)^2}\int_0^{\theta_{\text{max}}} d\theta\, \frac{1}{\theta}\left(|\hat{\mathcal{M}}^2|^2 + \left(\frac{E+E'}{E-E'}\right)^2 |\hat{\mathcal{M}}^1|^2\right)
\end{aligned} \tag{28}$$

where in the last step we used the fact that $q^4 \approx (-EE'\theta^2)^2$ in the small θ limit. Notice that the integral is logarithmically divergent for $\theta \to 0$.

(d) To treat the divergence we reintroduce the electron mass in the expression for q^2 just as we did for PS Problem 5.5,

$$q^2 = (p'-p)^2 = -2(EE' - pp'\cos\theta) + 2m^2\,, \tag{29}$$

where $p = |\mathbf{p}|$ and $p' = |\mathbf{p}'|$. Since $E, E' \gg m$, we can approximate

$$p = \sqrt{E^2 - m^2} \approx E - \frac{m^2}{2E} \qquad p' = \sqrt{E'^2 - m^2} \approx E' - \frac{m^2}{2E'}\,. \tag{30}$$

Plugging these into the expression for q^2 gives us, after some rearrangement,

$$q^2 \approx -EE'\left[\theta^2 + m^2\left(\frac{1}{E'} - \frac{1}{E}\right)^2\right]\,, \tag{31}$$

where we have kept only the leading terms. Substituting this q^2 into the integral for σ in Eq. (28), we find:

$$\sigma \approx \frac{e^2}{4E^2}\int d\Pi'_Y \int \frac{dE'}{(2\pi)^2}\int_0^{\theta_{\text{max}}} d\theta\, \frac{\theta^3}{\left[\theta^2 + m^2\left(\frac{1}{E'} - \frac{1}{E}\right)^2\right]^2}\left(|\hat{\mathcal{M}}^2|^2 + \left(\frac{E+E'}{E-E'}\right)^2 |\hat{\mathcal{M}}^1|^2\right)\,. \tag{32}$$

If we let $\frac{1}{\mathcal{E}} \equiv \left(\frac{1}{E'} - \frac{1}{E}\right)^2$, then the integral over θ can be evaluated as:

$$\begin{aligned}
\int_0^{\theta_{\text{max}}} d\theta\, \frac{\theta^3}{[\theta^2 + m^2/\mathcal{E}]^2} &= -\frac{1}{2}\log(m^2/\mathcal{E}) + \frac{1}{2}\log(\theta_{\text{max}} + m^2/\mathcal{E}) - \frac{\theta^2_{\text{max}}}{2(\theta^2_{\text{max}} + m^2/\mathcal{E})} \\
&\approx -\frac{1}{2}\log(m^2/\mathcal{E})
\end{aligned} \tag{33}$$

where we have approximated the integral by the dominant logarithmic singularity in the small m limit. First, we argue hand-wavingy: since $\mathcal{E}$ depends on the momentum transfer in the scattering process, and has dimensions of energy squared, we can say that $\mathcal{E} \sim \mathrm{O}(s)$, where the Mandelstam variable $s = (p+k)^2$ is the total initial four-momentum of the system. Thus $\log(m^2/\mathcal{E}) \approx -\log(s/m^2)$, and we can write the cross-section σ as:

$$\sigma \approx \frac{e^2}{8E^2}\int d\Pi'_Y \int \frac{dE'}{(2\pi)^2}\log\left(\frac{s}{m^2}\right)\left(|\hat{\mathcal{M}}^2|^2 + \left(\frac{E+E'}{E-E'}\right)^2 |\hat{\mathcal{M}}^1|^2\right)\,. \tag{34}$$

We will establish this result with more rigor in part (e), where we take the difference between $\mathcal{E}$ and s more seriously.

(e) We assume the target cross-sections are independent of the photon polarization, which implies $|\hat{\mathcal{M}}^1|^2 = |\hat{\mathcal{M}}^2|^2 \equiv |\hat{\mathcal{M}}|^2$. Let us introduce the variable $x = (E - E')/E$, the fraction of the initial electron energy carried off by the photon. Then

$$dx = -\frac{dE'}{E} \qquad \text{and} \qquad \left(\frac{E+E'}{E-E'}\right)^2 = \frac{(2-x)^2}{x^2}\,. \tag{35}$$

We can also express $\mathcal{E}$ with x and s

$$s = (E + E_X)^2 \approx 4E^2 + M_X^2 \approx 4E^2 \tag{36}$$

$$\mathcal{E} = \left(\frac{EE'}{E-E'}\right)^2 = \left(\frac{1-x}{x}\right)^2 E^2 \approx \left(\frac{1-x}{2x}\right)^2 s \tag{37}$$

where we assumed $E \gg M_X$. The cross-section of Eq. (34) with the proper treatment $\mathcal{E}$ becomes:

$$\begin{aligned}
\sigma &\approx \frac{\alpha}{8\pi E}\int_0^1 dx \log\left(\frac{\mathcal{E}}{m^2}\right)\left(1 + \frac{(2-x)^2}{x^2}\right)\int d\Pi'_Y |\hat{\mathcal{M}}|^2 \\
&\approx \frac{\alpha}{8\pi E}\int_0^1 dx \left[\log\left(\frac{s}{m^2}\right) + 2\log\left(\frac{1-x}{2x}\right)\right]\left(1 + \frac{(2-x)^2}{x^2}\right)\int d\Pi'_Y |\hat{\mathcal{M}}|^2 + ... \\
&\approx \frac{\alpha}{4\pi E}\int_0^1 dx \log\left(\frac{s}{m^2}\right)\frac{1+(1-x)^2}{x^2}\int d\Pi'_Y |\hat{\mathcal{M}}|^2\,,
\end{aligned} \tag{38}$$

where we dropped an $\mathcal{O}(1)$ term by neglecting $2\log\left(\frac{1-x}{2x}\right)$ in the integrand. Note that

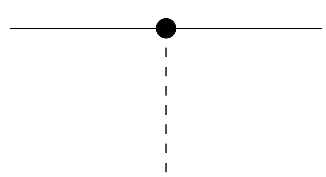
$$\frac{1}{2xE}\int d\Pi'_Y |\hat{\mathcal{M}}|^2 = \frac{1}{2E_X\, 2(xE)(1+v_X)}\int d\Pi_Y |\hat{\mathcal{M}}|^2 \equiv \sigma_{\gamma X}(x) \tag{39}$$

is the cross-section of a photon of incident energy xE hitting the target. Our total cross-section can be written:

$$\begin{aligned}
\sigma &\approx \frac{\alpha}{2\pi}\int_0^1 dx \log\left(\frac{s}{m^2}\right)\frac{1+(1-x)^2}{x}\sigma_{\gamma X}(x) \\
&= \int_0^1 dx\, N_\gamma(x)\sigma_{\gamma X}(x)
\end{aligned} \tag{40}$$

where $N_\gamma(x) = \frac{\alpha}{2\pi}\frac{1+(1-x)^2}{x}\log\left(\frac{s}{m^2}\right)$. Thus σ effectively looks like the cross-section of a beam of photons with energies xE and distribution $N_\gamma(x)$ hitting the target. $N_\gamma(x)$ is the probability to find a photon of longitudinal fraction x in the incident electron, when probing it in a collision with CM energy squared s.

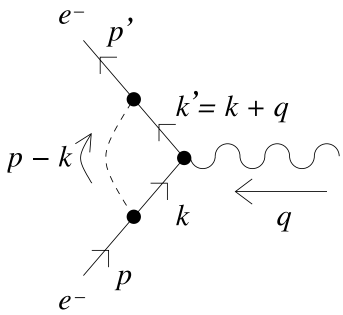
2. (a) We will represent the vertex corresponding to $H_{\text{int}} = \int d^3x\, \frac{\lambda}{\sqrt{2}} h\bar{\psi}\psi$, the coupling of the Higgs boson to the electron, by:



$$= \quad -i\frac{\lambda}{\sqrt{2}}\,.$$

To find the effect of the Higgs on the electron $g-2$, we look at its contribution to the electron-photon vertex,

We know that the total vertex has the form $\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)$, and that $g - 2 = 2F_2(0)$. At one loop we don't have to worry about divergences for $F_2(0)$, hence we can do the Dirac algebra in 4 dimensions and forget about dimensional regularization. Thus to find the Higgs contribution to $g-2$, we need to rewrite $\bar{u}(p')\delta\Gamma^\mu(p',p)u(p)$ so as to extract the coefficient of $\frac{i\sigma^{\mu\nu}q_\nu}{2m}$ and evaluate it at $q^2 = 0$.



$$= \quad -ie\,\bar{u}(p')\delta\Gamma^\mu(p',p)u(p)\,.$$

The full expression for $\bar{u}(p')\delta\Gamma^\mu(p',p)u(p)$ is:

$$\begin{aligned}
\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{(p-k)^2-m_h^2+i\epsilon}\bar{u}(p')\left(-\frac{i\lambda}{\sqrt{2}}\right)\frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon}\gamma^\mu \\
&\quad\cdot\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\left(-\frac{i\lambda}{\sqrt{2}}\right)u(p) \\
&= \frac{i\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}(p')(\not{k}'+m)\gamma^\mu(\not{k}+m)u(p)}{[(p-k)^2-m_h^2+i\epsilon][k'^2-m^2+i\epsilon][k^2-m^2+i\epsilon]} \\
&= \frac{i\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}(p')[\not{k}'\gamma^\mu\not{k}+m^2\gamma^\mu+m(\gamma^\mu\not{k}+\not{k}'\gamma^\mu)]u(p)}{[(p-k)^2-m_h^2+i\epsilon][k'^2-m^2+i\epsilon][k^2-m^2+i\epsilon]} \qquad (41)
\end{aligned}$$

The way we evaluate this expression is analogous to the algebra described in Section 6.3 of Peskin and Schroeder. We look individually at the denominator and numerator of the integrand. For the denominator, we let $A_1 = k^2 - m^2 + i\epsilon$, $A_2 = k'^2 - m^2 + i\epsilon$, $A_3 = (p-k)^2 - m_h^2 + i\epsilon$, and rewrite it using Feynman parameters as:

$$\frac{1}{A_1A_2A_3} = \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{D^3} \qquad (42)$$

where

$$\begin{aligned}
D &= xA_1 + yA_2 + zA_3 \\
&= x(k^2-m^2) + y(k'^2-m^2) + z[(k-p)^2-m_h^2] + (x+y+z)i\epsilon \\
&= k^2 + 2k\cdot(yq-zp) + yq^2 + zp^2 - (x+y)m^2 - zm_h^2 + i\epsilon \qquad (43)
\end{aligned}$$

In the second line we have used the fact that $x+y+z=1$ and $k' = k+q$. We shift k to complete the square, introducing a new variable $l = k + yq - zp$. With some work, using electron momenta that are on shell, D simplifies to $D = l^2 - \Delta + i\epsilon$, where $\Delta = -xyq^2 + (1-z)^2m^2 + zm_h^2$.
Turning to the numerator, we would like to express it in terms of l. To simplify the algebra, we recall the useful results,

$$\int\frac{d^4l}{(2\pi)^4}\frac{l^\mu}{D^3} = 0, \qquad \int\frac{d^4l}{(2\pi)^4}\frac{l^\mu l^\nu}{D^3} = \int\frac{d^4l}{(2\pi)^4}\frac{\frac{1}{4}g^{\mu\nu}l^2}{D^3}\,, \qquad (44)$$

which allow our numerator to be written in the equivalent form

$$\begin{aligned}
\text{numerator} &= \bar{u}(p')[\not{k}'\gamma^\mu\not{k}+m^2\gamma^\mu+m(\gamma^\mu\not{k}+\not{k}'\gamma^\mu)]u(p) \\
&\to \bar{u}(p')\Big[-\frac{1}{2}\gamma^\mu l^2 + ((1-y)\not{q}+z\not{p})\gamma^\mu(-y\not{q}+z\not{p}) + m^2\gamma^\mu \\
&\quad + m[\gamma^\mu(-y\not{q}+z\not{p}) + ((1-y)\not{q}+z\not{p})\gamma^\mu]\Big]u(p)\,. \qquad (45)
\end{aligned}$$

This can be further simplified using the identities

$$\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu\not{p}\,, \qquad \not{p}u(p) = mu(p)\,, \qquad \bar{u}(p')\not{p}' = \bar{u}(p')m\,, \qquad q^2 = -2p'\cdot p + 2m^2\,. \qquad (46)$$

The terms involving slashed variables can be rewritten:

$$\begin{aligned}\bar{u}(p')((1-y)\not{q}+z\not{p})\gamma^\mu(-y\not{q}+z\not{p})u(p) &= \bar{u}(p')[\gamma^\mu(q^2xy-m^2z(z-2))-2mz(xp^\mu+yp'^\mu)]u(p)\\ &= \bar{u}(p')[\gamma^\mu(q^2xy-m^2z(z-2))-mz(p+p')^\mu(x+y)\\ &\quad +mzq^\mu(x-y)]u(p) \qquad (47)\\ \bar{u}(p')m[\gamma^\mu(-y\not{q}+z\not{p})+((1-y)\not{q}+z\not{p})\gamma^\mu]u(p) &= \bar{u}(p')[2m^2\gamma^\mu-2m(xp^\mu+yp'^\mu)]u(p)\\ &= \bar{u}(p')[2m^2\gamma^\mu-m(p+p')^\mu(x+y)+mq^\mu(x-y)]u(p)\,. \qquad (48)\end{aligned}$$

Plugging Eqs. (47) and (48) into Eq. (45), our numerator becomes:

$$\begin{aligned}\text{numerator} &= \bar{u}(p')\Big[\gamma^\mu(-\frac{1}{2}l^2+q^2xy+m^2(3+2z-z^2))-m(p+p')^\mu(x+y)(1+z)\\ &\quad +mq^\mu(x-y)(1+z)\Big]u(p)\\ &= \bar{u}(p')\Big[\gamma^\mu(-\frac{1}{2}l^2+q^2xy+m^2(1+z)^2)+\frac{i\sigma^{\mu\nu}q_\nu}{2m}2m^2(1-z^2)\\ &\quad +mq^\mu(x-y)(1+z)\Big]u(p)\,. \qquad (49)\end{aligned}$$

In the second line we have used the Gordon identity and the fact that $x+y+z=1$. Note that the term proportional to q^μ is odd under exchange of x and y and will integrate to zero, which is required by the gauge invariance. The part we are interested in is the coefficient of $\frac{i\sigma^{\mu\nu}q_\nu}{2m}$, which gives us the contribution to F_2. Putting this together with the results for the denominator, we have:

$$\begin{aligned}\delta F_2(q^2) &= i\lambda^2\int\frac{d^4l}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2m^2(1-z^2)}{[l^2-\Delta+i\epsilon]^3}\\ &= \frac{\lambda^2}{2(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2m^2(1-z^2)}{\Delta}\\ &= \frac{\lambda^2}{2(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2m^2(1-z^2)}{-xyq^2+(1-z)^2m^2+zm_h^2} \qquad (50)\end{aligned}$$

For $q^2=0$, we can evaluate the x and y integrals:

$$\begin{aligned}\delta F_2(0) &= \frac{\lambda^2}{2(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2m^2(1-z^2)}{(1-z)^2m^2+zm_h^2}\\ &= \frac{\lambda^2}{2(4\pi)^2}\int_0^1 dz\int_0^{1-z} dy\,\frac{2m^2(1-z^2)}{(1-z)^2m^2+zm_h^2}\\ &= \frac{\lambda^2}{2(4\pi)^2}\int_0^1 dz\,\frac{2m^2(1-z^2)(1-z)}{(1-z)^2m^2+zm_h^2}\\ &= \frac{\lambda^2}{(4\pi)^2}\frac{m^2}{m_h^2}\int_0^1 dz\,\frac{(1-z^2)(1-z)}{z+(1-z)^2\frac{m^2}{m_h^2}} \qquad (51)\end{aligned}$$

This is a very ugly integral, although Mathematica can do it. Note that since $m \ll m_h$, we might think about expanding the integrand and integrating term by term. The leading order term in that expansion is, however, singular (there is an IR divergence). It's still much nicer, with very small loss in accuracy to express the result as an expansion in m/m_h. I find best way to do this is to evaluate the integral in Mathematica with the assumptions $\mathcal{M}\equiv m^2/m_h^2>0$ and $4\mathcal{M}<1$. Then take the result and series expand around $\mathcal{M}=0$. This gives

$$\delta F_2(0) = \frac{\lambda^2}{(4\pi)^2}\frac{m^2}{m_h^2}\left[\log\frac{m_h^2}{m^2}-\frac{7}{6}+\mathrm{O}\left(\frac{m^2}{m_h^2}\right)\right] \qquad (52)$$

This is the Higgs contribution to $a_h=\frac{g-2}{2}=F_2(0)$.

(b) We need to check whether the values $\lambda = 3\times 10^{-6}$ and $m_h > 60$ GeV are consistent with the experimental constraint $|a_h| < 1\times 10^{-10}$. Plugging into Eq. (52), we find

$$|a_h| < 9\times 10^{-23} \qquad \text{for } m = 0.5110 \text{ MeV}, \ \lambda = 3\times 10^{-6}, \ m_h > 60 \text{ GeV} \tag{53}$$

so these values for λ and m_h are not excluded. ($|a_h|$ is a monoton decreasing function of m_h.)
For the muon, $\lambda = 6\times 10^{-4}$, and the experimental constraint is $|a_h| < 3\times 10^{-8}$. Plugging into Eq. (52), we find

$$|a_h| < 8\times 10^{-14} \qquad \text{for } m = 105.6 \text{ MeV}, \ \lambda = 6\times 10^{-4}, \ m_h > 60 \text{ GeV} \tag{54}$$

so these values for λ and m_h are also not excluded.

(c) For the axion, which couples to the electron according to $H_{\text{int}} = \int d^3x\, \frac{i\lambda}{\sqrt{2}} a\bar{\psi}\gamma^5\psi$, we do a calculation analogous to the one in part a). The axion contribution to the electron-photon vertex looks like:

$$\begin{aligned}
\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) &= \int \frac{d^4k}{(2\pi)^4} \frac{i}{(p-k)^2 - m_a^2 + i\epsilon} \bar{u}(p') \left(\frac{\lambda\gamma^5}{\sqrt{2}}\right) \frac{i(\not{k}' + m)}{k'^2 - m^2 + i\epsilon} \gamma^\mu \\
&\quad \cdot \frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon} \left(\frac{\lambda\gamma^5}{\sqrt{2}}\right) u(p) \\
&= \frac{i\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-\bar{u}(p')\gamma^5(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^5 u(p)}{[(p-k)^2 - m_a^2 + i\epsilon][k'^2 - m^2 + i\epsilon][k^2 - m^2 + i\epsilon]} \\
&= \frac{i\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-\bar{u}(p')\gamma^5[\not{k}'\gamma^\mu\not{k} + m^2\gamma^\mu + m(\gamma^\mu\not{k} + \not{k}'\gamma^\mu)]\gamma^5 u(p)}{[(p-k)^2 - m_a^2 + i\epsilon][k'^2 - m^2 + i\epsilon][k^2 - m^2 + i\epsilon]}
\end{aligned} \tag{55}$$

The denominator is identical to the one from part a), but the numerator differs by a minus sign and extra factors of γ^5. Since γ^5 anticommutes with γ^μ, and $\gamma^5\gamma^5 = 1$, we can rewrite the numerator as:

$$\begin{aligned}
\text{numerator} &= -\bar{u}(p')\gamma^5[\not{k}'\gamma^\mu\not{k} + m^2\gamma^\mu + m(\gamma^\mu\not{k} + \not{k}'\gamma^\mu)]\gamma^5 u(p) \\
&= \bar{u}(p')[\not{k}'\gamma^\mu\not{k} + m^2\gamma^\mu - m(\gamma^\mu\not{k} + \not{k}'\gamma^\mu)]u(p) \\
&\to \bar{u}(p')\Big[-\frac{1}{2}\gamma^\mu l^2 + ((1-y)\not{q} + z\not{p})\gamma^\mu(-y\not{q} + z\not{p}) + m^2\gamma^\mu \\
&\quad - m[\gamma^\mu(-y\not{q} + z\not{p}) + ((1-y)\not{q} + z\not{p})\gamma^\mu]\Big] u(p)\,.
\end{aligned} \tag{56}$$

Plugging Eqs. (47) and (48) into Eq. (56), our numerator becomes:

$$\begin{aligned}
\text{numerator} &= \bar{u}(p')\Big[\gamma^\mu(-\frac{1}{2}l^2 + q^2xy + m^2(-1 + 2z - z^2)) - m(p+p')^\mu(x+y)(z-1) \\
&\quad + mq^\mu(x-y)(z-1)\Big]u(p) \\
&= \bar{u}(p')\Big[\gamma^\mu(-\frac{1}{2}l^2 + q^2xy + m^2(1-z)^2) - \frac{i\sigma^{\mu\nu}q_\nu}{2m} 2m^2(1-z)^2 \\
&\quad + mq^\mu(x-y)(z-1)\Big]u(p)
\end{aligned} \tag{57}$$

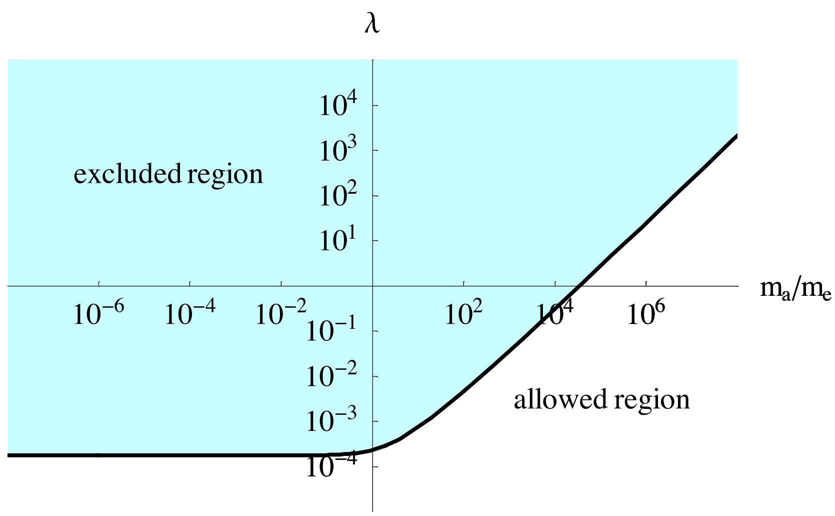
As above, the part we are interested in is the coefficient of $\frac{i\sigma^{\mu\nu}q_\nu}{2m}$, which gives us the contribution to F_2. Putting this together with the results for the denominator, we have:

$$\begin{aligned}
\delta F_2(q^2) &= -i\lambda^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2m^2(1-z)^2}{[l^2 - \Delta + i\epsilon]^3} \\
&= -\frac{\lambda^2}{2(4\pi)^2} \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2m^2(1-z)^2}{\Delta} \\
&= -\frac{\lambda^2}{2(4\pi)^2} \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2m^2(1-z)^2}{-xyq^2 + (1-z)^2m^2 + zm_a^2}
\end{aligned} \tag{58}$$

For $q^2 = 0$, we can evaluate the x and y integrals:

$$\begin{aligned}
\delta F_2(0) &= -\frac{\lambda^2}{2(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2m^2(1-z)^2}{(1-z)^2m^2+zm_a^2} \\
&= -\frac{\lambda^2}{2(4\pi)^2}\int_0^1 dz\int_0^{1-z} dy\,\frac{2m^2(1-z)^2}{(1-z)^2m^2+zm_a^2} \\
&= -\frac{\lambda^2}{(4\pi)^2}\int_0^1 dz\,\frac{m^2(1-z)^3}{(1-z)^2m^2+zm_a^2} \\
&\equiv I(\lambda, m_a)
\end{aligned} \tag{59}$$

We can numerically evaluate $I(\lambda, m_a)$ to find the allowed region where $|I(\lambda, m_a)| < 10^{-10}$. The results are shown below, as a log-log plot:



Here is a sample Mathematica command to generate the plot:

```
table = Table[{Exp[M], λ/.
NSolve[λ^2/(4 * Pi)^2 * NIntegrate[(1 − x)^3/((1 − x)^2 + Exp[2M] * x), {x, 0, 1}]
 == 10^(−10), λ][[2]]}, {M, −15, 15, 0.03}];
ListLogLogPlot[table, Joined → True, AxesOrigin → {1, 1}]
```

3. (a) The fermion self energy contribution due to Yukawa interaction is given by (we use dimensional regularization)

$$\begin{aligned}
-i\Sigma_2^{\phi}(\not p) &= \left(-i\frac{\lambda}{\sqrt{2}}\right)^2\int\frac{d^Dk}{(2\pi)^D}\,\frac{i(\not k+m_e)}{k^2-m_e^2}\,\frac{i}{(p-k)^2-m_h^2} \\
&= i\frac{\lambda^2}{2}\int\frac{d^Dl_E}{(2\pi)^D}\int_0^1 dz\frac{m_e+z\not p}{(l_E^2+\Delta_1+i\epsilon)^2},
\end{aligned} \tag{60}$$

where in the numerator we have dropped the linear term in $l = k - zp$ and $\Delta_1 = (1-z)m_e^2 + zm_h^2 - z(1-z)p^2$. Performing the integral over l gives

$$\begin{aligned}
\Sigma_2^{\phi}(\not p) &= -\frac{\lambda^2}{32\pi^2}\int_0^1 dz\,(m_e+z\not p)\left(\frac{4\pi\mu^2}{\Delta_1}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right) \\
&= -\frac{\lambda^2}{32\pi^2}\int_0^1 dz\,(m_e+z\not p)\left(\frac{2}{\epsilon}-\gamma+\log\left(\frac{4\pi\mu^2}{\Delta_1}\right)\right)
\end{aligned} \tag{61}$$

Then we have

$$\delta_\phi Z_2 = \frac{d\Sigma_2^{\phi}}{d\not p}|_{\not p=m_e} = -\frac{\lambda^2}{32\pi^2}\int_0^1 dz\left[z\left(\frac{2}{\epsilon}-\gamma+\log\left(\frac{4\pi\mu^2}{\Delta_2}\right)\right)+\frac{2z(1-z)(1+z)m_e^2}{\Delta_2}\right] \tag{62}$$

where $\Delta_2 = \Delta_1(p^2 = m_e^2) = (1-z)^2 m_e^2 + z m_h^2$.
For Z_1, we have to compute the contribution to the vertex with an exchange of a Higgs. Fortunately, we have already done most of the work in Problem 2. Copying the numerator form (49), the denominator from (50) and setting $q = 0$ we get

$$\begin{aligned}\bar{u}(p')\delta_\phi\Gamma^\mu u(p) &= -i\lambda^2 \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \frac{\bar{u}(p)\gamma^\mu \left(-\frac{D-2}{D}\; l^2 + m^2(1+z)^2\right) u(p)}{[l^2 - \Delta_2]^3} \\ &= \bar{u}(p)\gamma^\mu u(p) \int_0^1 dz(1-z) \int \frac{d^D l_E}{(2\pi)^D} \frac{\lambda^2}{(l_E^2 + \Delta_2)^3} \left(\frac{D-2}{D}\; l_E^2 + (1+z)^2 m_e^2\right)\end{aligned}$$

where once again $\Delta_2 = (1-z)^2 m_e^2 + z m_h^2$ and we payed attention to D when doing the Dirac algebra unlike in Problem 2, where we were calculating a finite quantity. Performing the integral over l_E and taking $D = 4 - \epsilon$

$$\delta_\phi Z_1 = -\frac{\lambda^2}{32\pi^2} \int_0^1 dz(1-z) \left[\frac{2}{\epsilon} - 1 - \gamma + \log\left(\frac{4\pi\mu^2}{\Delta_2}\right) + \frac{(1+z)^2 m_e^2}{\Delta_2}\right] \tag{63}$$

Then the difference is

$$\delta_\phi(Z_2 - Z_1) = \frac{\lambda^2}{32\pi^2} \int_0^1 dz \big[(1-2z)\left[\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi\mu^2}{\Delta_2}\right)\right] - (1-z) + m_e^2 \frac{(1+z)(1-z)^2}{\Delta_2}\big] \tag{64}$$

Integrating by parts (the boundary terms vanish), we get

$$\int_0^1 dz(1-2z)\log\left(\frac{\mu^2}{\Delta_2}\right) = \int_0^1 dz\; z(1-z) \frac{-2(1-z)m_e^2 + m_h^2}{\Delta_2} \tag{65}$$

Thus

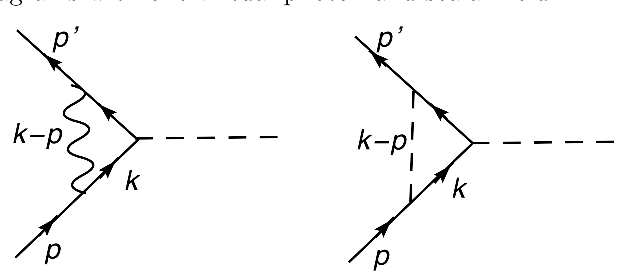
$$\delta_\phi(Z_2 - Z_1) = \frac{\lambda^2}{32\pi^2} \int_0^1 dz \frac{1}{\Delta_2}\left[-2z(1-z)^2 m_e^2 + z(1-z)m_h^2 - (1-z)\Delta_2 + m_e^2(1+z)(1-z)^2\right] = 0 \tag{66}$$

We can easily see that this result combines to zero. The Ward identity will be true in this theory. This can be seen in many different ways. Firstly, the Yukawa term behaves the same way as a fermion mass term under gauge transformation (assuming that ϕ is not charged under $U(1)$), hence the gauge invariance is maintained. The Ward identity is the consequence of gauge symmetry, hence it must be true with this new interaction. Secondly, we note that the addition of a fermion-fermion-scalar vertex does not affect the diagrammatic proof of the Ward identity at all, since the photon does not couple to ϕ.

(b) By analogy with the QED electron vertex $\Gamma^\mu(p,k)$ we define the exact Yukawa vertex with two on-shell electron lines and one off-shell scalar line as

$$\bar{u}(p')\left[-i\frac{\lambda}{\sqrt{2}}\Gamma(p',p)\right]u(p) \tag{67}$$

and define the renormalization factor as $Z_y^{-1} = \Gamma(q=0)$. At tree level $Z_y = 1$, at one loop level δZ^{-1} gets contributions from the diagrams with one virtual photon and scalar field:



$$\begin{aligned}\bar{u}(p')\delta\Gamma(q=0)u(p) = \int \frac{d^4k}{(2\pi)^4} &\left[\bar{u}(p')\frac{-ig_{\mu\nu}}{(k-p)^2 + i\epsilon - \mu^2}(-ie\gamma^\nu)\frac{i(\not{k}+m)}{k^2 - m^2 + i\epsilon}\frac{i(\not{k}+m)}{k^2 - m^2 + i\epsilon}(-ie\gamma^\mu)u(p)\right. \\ &\left. +\bar{u}(p')\frac{i}{(k-p)^2 - m_h^2 + i\epsilon}\left(-\frac{i\lambda}{\sqrt{2}}\right)\frac{i(\not{k}+m)}{k^2 - m^2 + i\epsilon}\frac{i(\not{k}+m)}{k^2 - m^2 + i\epsilon}\left(-\frac{i\lambda}{\sqrt{2}}\right)u(p)\right]\end{aligned} \tag{68}$$

Making use of the Feynmann parameters

$$\delta Z_y = \int_0^1 dz(1-z) \int \frac{d^4 l_E}{(2\pi)^4} \Big[2e^2 \frac{\gamma^\nu (\not{l}_E + z\not{p} + m)^2 \gamma_\nu}{(l_E^2 + \Delta_1)^3} - \frac{\lambda^2 (\not{l}_E + z\not{p} + m)^2}{(l_E^2 + \Delta_2)^3} \Big] \tag{69}$$

where

$$\Delta_1 = (1-z)^2 m^2 + z\mu^2 \tag{70}$$

$$\Delta_2 = (1-z)^2 m^2 + z m_h^2 \tag{71}$$

We are instructed to calculate just the UV divergent parts, hence we only keep the l_E^2 terms in the numerator. Hence, the integrals over the momenta and z give

$$\delta Z_y = \left(-\frac{\alpha}{\pi} + \frac{\lambda^2}{32\pi^2} \right) \frac{2}{\epsilon} + \text{finite terms.} \tag{72}$$

Using the result obtained for the electron self energy diagram in part (a) and the photon contribution from class

$$\delta Z_2 = -\left(\frac{\alpha}{4\pi} + \frac{\lambda^2}{64\pi^2} \right) \frac{2}{\epsilon} \tag{73}$$

Thus $\delta Z_y \neq \delta Z_2$. Note however, that by including a counter term for the Yukawa interaction (and for every possible term in the Lagrangian allowed by symmetry) all the divergences can be made to cancel. $Z_2 \neq Z_y$ has nothing to do with gauge symmetry, which is represented by $Z_1 = Z_2$ and hence there is no pathology (i.e. anomalous symmetry) in this theory.

8.324 Relativistic Quantum Field Theory II
Fall 2010