Lecture 21 (Nov. 22, 2017)

21.1 SO(3) versus SU(2)

As we saw, if we study only the representations of SO(3), we will only find representations with integer spin. However, by considering the Lie algebra $\mathfrak{so}(3) \cong \mathfrak{su}(2)$, we found representations with half-integer spin. These should properly be considered as representations of SU(2), which is a double cover of SO(3), but has the same Lie algebra. Linear representations of SU(2) include both integer and half-integer spins, while linear representations of SO(3) include only integer spins. As representations of SO(3), the half-integer spin representations are what are called *projective* representations, meaning that they satisfy the group structure only up to a phase:

$$\mathcal{D}(R_1)\mathcal{D}(R_2) = e^{i\xi(R_1,R_2)}\mathcal{D}(R_1 R_2) \,. \tag{21.1}$$

The elements of SU(2) are in two-to-one correspondence with the elements of SU(3), which is why we say that SU(2) is a double cover of SO(3). The two SU(2) matrices U and $-U$ both correspond to the same SO(3) matrix R. This can be seen directly from the expression

$$R_{ij} = \frac{1}{2}\operatorname{tr}\!\left(\sigma_j U^\dagger \sigma_i U\right), \tag{21.2}$$

which expresses a 3×3 rotation matrix R in terms of a 2×2 unitary matrix U. On the homework, you will prove that R defined in this way is a proper rotation matrix.

21.2 Addition of Angular Momentum

Given two systems 1 and 2, with angular momenta j_1 and j_2, respectively, what are the possible total angular momenta for the combined system? We first must choose a basis for the combined Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. We will choose the basis

$$|j_1, m_1; j_2, m_2\rangle = |j_1.m_1\rangle \otimes |j_2, m_2\rangle \,. \tag{21.3}$$

We know how to write down operators on this tensor product Hilbert space. For example, the total angular momentum is

$$\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2 := \boldsymbol{J}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \boldsymbol{J}_2 \,, \tag{21.4}$$

where the rightmost expression makes the meaning completely clear, and the middle expression is a convenient shorthand that suppresses the products with the identity. Note that this combined operator satisfies the angular momentum commutation algebra, $[J_i, J_j] = i\epsilon_{ijk}J_k$. (If we took the difference of two angular momenta, rather than the sum, then the result would *not* satisfy the angular momentum commutation algebra.) Note also that

$$\left[J_1^i, J_2^j\right] = 0 \,, \tag{21.5}$$

because the spin operators on one subsystem act trivially on the other subsystem.

We have

$$\boldsymbol{J}^2 = \boldsymbol{J}_1^2 + \boldsymbol{J}_2^2 + 2\boldsymbol{J}_1 \cdot \boldsymbol{J}_2 \,, \tag{21.6}$$

from which we see that

$$\left[\boldsymbol{J}^2, \boldsymbol{J}_{a,z}\right] \neq 0 \,, \tag{21.7}$$

where $a = 1, 2$. Thus, we see that the eigenvalue of $\boldsymbol{J}^2$ is not a good quantum number in the basis we have chosen, in the sense that the basis we have chosen is not an eigenbasis of $\boldsymbol{J}^2$.

However, because $\boldsymbol{J}$ satisfies the angular momentum commutation algebra, we know that $\boldsymbol{J}^2$ and J_z can be simultaneously diagonalized. What are the allowed eigenvalues of $\left(\boldsymbol{J}^2, J_z\right)$, given J_1 and J_2? As an example, if we consider two spin-$\frac{1}{2}$ systems, there are four states in the combined Hilbert space, which can be written in the form

$$\left|j_1 = \frac{1}{2}, m_1, j_2 = \frac{1}{2}, m_2\right\rangle = |++\rangle, |+-\rangle, |-+\rangle, |--\rangle. \tag{21.8}$$

Clearly,

$$J_z|++\rangle = |++\rangle, \tag{21.9}$$

which implies that $m = +1$ for $|++\rangle$. Similarly, we find $m = -1$ for $|--\rangle$, and $m = 0$ for both $|+-\rangle$ and $|-+\rangle$. These organize into a singlet $j = 0$,

$$\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle), \tag{21.10}$$

and a triplet $j = 1$,

$$|++\rangle,$$
$$\frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle), \tag{21.11}$$
$$|--\rangle.$$

In general, we want to know what the coefficients are for the change of basis from $|j_1, m_1; j_2, m_2\rangle$ to the new basis $|j, m; j_1, j_2\rangle$, i.e., the coefficients

$$\langle j_1, m_1; j_2, m_2 | j, m; j_1, j_2\rangle. \tag{21.12}$$

These are known as *Clebsch–Gordon coefficients*. We now discuss some general properties of Clebsch–Gordon coefficients:

1. The Clebsch–Gordon coefficients are zero unless $m = m_1 + m_2$. To see this, note that

$$J_z - J_{1,z} - J_{2,z} = 0 \tag{21.13}$$

   as an operator. In particular,

$$(J_z - J_{1,z} - J_{2,z})|j, m; j_1, j_2\rangle = 0. \tag{21.14}$$

   Then,

$$\langle j_1, m_1; j_2, m_2 | (J_z - J_{1,z} - J_{2,z}) | j, m; j_1, j_2\rangle = 0. \tag{21.15}$$

   Acting with J_z to the right and with $J_{i,z}$ to the left, this gives us

$$(m - m_1 - m_2)\langle j_1, m_1; j_2, m_2 | j, m; j_1, j_2\rangle = 0. \tag{21.16}$$

   Thus, the Clebsch–Gordon coefficient must vanish unless $m = m_1 + m_2$.

2. The Clebsch–Gordon coefficients vanish unless $|j_1 - j_2| \le j \le j_1 + j_2$. As a consistency check, we can see that this condition gives the correct number of states. In the basis $|j_1, m_1; j_2.m_2\rangle$, we can see that the total number of states is

$$(2j_1 + 1)(2j_2 + 1), \tag{21.17}$$

because subsystem i has $2j_i + 1$ possible states. We can similarly count the number of states in the basis $|j, m; j_1, j_2\rangle$, assuming that $|j_1 - j_2| \leq j \leq j_1 + j_2$. For each j, there are $2j + 1$ states, and j runs between $|j_1 - j_2|$ and $j_1 + j_2$. We further assume that every j in this range occurs, but with two successive j values differing by 1 and not by $\frac{1}{2}$. This gives the total number of states as

$$\sum_{j=j_1-j_2}^{j_1+j_2} (2j + 1) = (2j_1 + 1)(2j_2 + 1) \,, \tag{21.18}$$

where we have assumed $j_1 \geq j_2$, without loss of generality.

3. If j_1 and j_2 are both half-integer (by which we mean an integer plus $\frac{1}{2}$) or both integer, then j is integer. To see this, note that under a 2π rotation, both subsystems will acquire the same phase (-1 if they are both half-integer, or $+1$ if they are both integer), and so the total system acquires no phase. Thus, the combined system has integer angular momentum.

   Using the same type of argument, we see that the converse is also true: if only one of (j_1, j_2) is half-integer, then j is half-integer.

21.3   Discrete Symmetries

We now move on to discuss parity (a.k.a. inversion). Let Π be the operator that implements the parity transformation. By this, we mean that

$$|\alpha\rangle \to \Pi|\alpha\rangle := |\alpha_\Pi\rangle \tag{21.19}$$

under the parity transformation $\boldsymbol{x} \to -\boldsymbol{x}$. We require that

$$\langle \alpha_\Pi | \boldsymbol{x} | \alpha_\Pi \rangle = -\langle \alpha | \boldsymbol{x} | \beta \rangle \,, \tag{21.20}$$

which implies that

$$\Pi^\dagger \boldsymbol{x} \Pi = -\boldsymbol{x} \,. \tag{21.21}$$

In other words,

$$\{\boldsymbol{x}, \Pi\} = 0 \,, \tag{21.22}$$

where $\{\cdot, \cdot\}$ is the anticommutator.

Two successive inversions do nothing, i.e., $\Pi^2 = \mathbb{1}$. This implies that the eigenvalues of Π are ± 1. States with eigenvalue $+1$ are called *even* under parity, and states with eigenvalue -1 are called *odd*. Operators that are odd under parity must anticommute with Π. For example, $\boldsymbol{p}$ is odd, as $\boldsymbol{p} \to -\boldsymbol{p}$ under parity, so

$$\{\boldsymbol{p}, \Pi\} = 0 \,. \tag{21.23}$$

What about angular momentum? For a general rotation (not necessarily a proper rotation),

$$\Pi R(\theta, \hat{\boldsymbol{n}}) = R(\theta, \hat{\boldsymbol{n}}) \Pi \,. \tag{21.24}$$

That is, parity and rotation commute with one another. This implies that $[\Pi, \boldsymbol{J}] = 0$. In general, $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$. We can see that $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$ is even under parity, because $\boldsymbol{x}$ and $\boldsymbol{p}$ are each odd. Because $\boldsymbol{J}$ and $\boldsymbol{L}$ are both even under parity, we conclude that $\boldsymbol{S}$ must also be even under parity.

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