Suppl. notes, part III   8.311, 2004

Power spectrum of radiation

consider charge initially & finally at rest: $r(t) \to 0$, $t \to \pm\infty$

$$f_{c,s}(t) = \sqrt{\frac{8\pi}{V}}\, e(\vec{v}\cdot\vec{e}_\lambda) \begin{cases} \cos \vec{k}\,\vec{r}(t) \\ \sin \vec{k}\,\vec{r}(t) \end{cases}$$

$$q_{c,s}(t) = \int_{-\infty}^{t} \frac{\sin\omega(t-t')}{\omega} f_{c,s}(t')\,dt'$$

$$q_{c,s}(t) = \frac{1}{2i\omega}\left(e^{i\omega t}\int_{-\infty}^{t} e^{-i\omega t'} f_{c,s}(t')\,dt' - e^{-i\omega t}\int_{-\infty}^{t} e^{i\omega t'} f_{c,s}(t')\,dt'\right)$$

define $q_\pm(t) = q_c \pm i q_s$

$$q_\pm(t) = \sqrt{\frac{2\pi}{V}}\frac{e}{i\omega}\Big(e^{i\omega t}\underbrace{\int_{-\infty}^{t} e^{-i\omega t' \pm i\vec{k}\vec{r}(t')}(\vec{v}(t')\cdot\vec{e}_\lambda)\,dt'}_{A_{\pm k}(\omega)} - e^{-i\omega t}\underbrace{\int_{-\infty}^{t} e^{i\omega t' \pm i\vec{k}\vec{r}(t')}(\vec{v}(t')\vec{e}_\lambda)\,dt'}_{A_{\pm k}(-\omega)}\Big)$$

$$A^*_{\vec{k}}(\omega) = A_{-\vec{k}}(-\omega)$$

@ large t $\int_{-\infty}^{t}\ldots \to \int_{-\infty}^{\infty}\ldots$

Energy: $\Delta E_c + \Delta E_s = \frac{1}{2}\dot{q}_c^2 + \frac{\omega^2}{2}q_c^2 + \frac{1}{2}\dot{q}_s^2 + \frac{\omega^2}{2}q_s^2 =$

$$= \frac{1}{4}\left(|\dot{q}_+|^2 + |\dot{q}_-|^2\right) + \frac{\omega^2}{4}\left(|q_+|^2 + |q_-|^2\right)$$

$$\Delta E_c + \Delta E_s = \frac{2\pi}{V}\frac{e^2}{\omega^2}\frac{\omega^2}{4}\left(|A_+(\omega)|^2 + |A_+(-\omega)|^2 + |A_-(\omega)|^2 + |A_-(-\omega)|^2\right)\times 2$$

$$E_{total} = V\int \frac{d^3k}{(2\pi)^3}\frac{2\pi}{V}e^2 \sum_{\vec{e}_\lambda} |A_{\vec{k}}(\omega)|^2,$$

$|A_{\pm\vec{k}}|^2$ correspond to radiation along $\vec{k}$ & $-\vec{k}$

$$\sum_{e_\lambda}(\vec{a}\vec{e}_\lambda)^2 = (\vec{a}\times\vec{n})^2, \quad \vec{n} = \frac{\vec{k}}{|k|}$$

$$E_{total} = \int \frac{d^3k}{(2\pi)^2}\left|e\int_{-\infty}^{\infty}\vec{v}(t)\times\vec{n}\, e^{i\vec{k}\vec{r}(t) - i\omega t}\,dt\right|^2$$

$$\frac{dE}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2 c^3}\left|\int_{-\infty}^{\infty}\vec{v}(t)\times\vec{n}\, e^{i\vec{k}\vec{r}(t) - i\omega t}\,dt\right|^2$$

Ex. 1 Let $\vec{v}(t) = \begin{cases} \vec{v}_1, & t<0 \\ \vec{v}_2, & t>0 \end{cases}$ $\qquad \vec{r}(t) = \begin{cases} \vec{v}_1 t, & t<0 \\ \vec{v}_2 t, & t>0 \end{cases}$

$$\int_0^\infty e^{i\alpha x}\,dx = \frac{1}{-i\alpha}, \qquad \int_{-\infty}^0 e^{i\alpha x}\,dx = \frac{1}{i\alpha}$$

$$\vec{k}\cdot\vec{r}(t) - \omega t = -\omega\left(1 - \vec{n}\cdot\vec{v}/c\right)t$$

$$\frac{dE}{d\Omega\, d\omega} = \frac{e^2}{4\pi^2 c^3}\left|\frac{\vec{v}_1\times\vec{n}}{1-\vec{v}_1\cdot\vec{n}/c} - \frac{\vec{v}_2\times\vec{n}}{1-\vec{v}_2\cdot\vec{n}/c}\right|^2$$

valid at low $\omega < \frac{2\pi}{\tau}$

$\uparrow$ collision time

Ex. 2 non-relativistic electron:

$$\frac{dE}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2 c^3}\left|\int \vec{v}(t)\times\vec{n}\, e^{-i\omega t}\,dt\right|^2$$

Ex. 3 non-relativistic collision

$$v(t) = \begin{cases} -v, & t<-\tau \\ vt/\tau, & -\tau<t<\tau \\ v, & t>\tau \end{cases}$$

$$\int_0^\tau + \int_\tau^\infty \left(\vec{v}\times\vec{n}\, e^{-i\omega t}\right)dt = \frac{\vec{v}\times\vec{n}}{\tau}\int_0^\tau t\, e^{-i\omega t}\,dt + \vec{v}\times\vec{n}\int_\tau^\infty e^{-i\omega t}\,dt$$

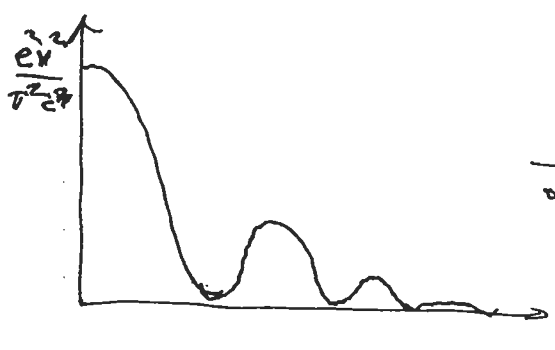
$$\int_0^\tau t\, e^{-i\omega t}\,dt = i\frac{\partial}{\partial\omega}\int_0^\tau e^{-i\omega t}\,dt = i\frac{\partial}{\partial\omega}\,\frac{1-e^{-i\omega\tau}}{i\omega} = \frac{e^{-i\omega\tau}-1}{\omega^2} - \frac{\tau e^{-i\omega\tau}}{i\omega}$$

$$\vec{v}\times\vec{n}\int_\tau^\infty e^{-i\omega t}\,dt = \frac{v\times n}{i\omega}e^{-i\omega\tau}$$

cancel

$$\left|\int_{-\infty}^\infty (v\times n)e^{-i\omega t}\,dt\right|^2 = \frac{(v\times n)^2}{\tau^2\omega^4}4$$

$$\frac{dE}{d\Omega\, d\omega} = \frac{e^2}{\pi^2 c^3}(\vec{v}\times\vec{n})^2\,\frac{\sin^2\omega\tau}{\omega^2\tau^2}$$



$\longrightarrow$ other $v(t)$