Problem Set #8

Problem 1 a) Charge conservation: $\rho L = \rho' L'$

Lorentz contraction $L' = \sqrt{1-\frac{v^2}{c^2}}\,L \;\rightarrow\; \rho' = \beta\rho$, $\beta = \left(1-\frac{v^2}{c^2}\right)^{-1/2}$

$j' = \rho' v = \beta v \rho$ $\rightarrow$ consistent with

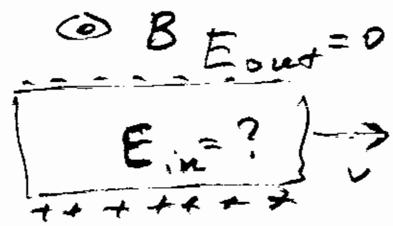
$\rho' = \beta(\rho + \frac{v}{c^2} j)$, $j' = \beta(j + v\rho)$

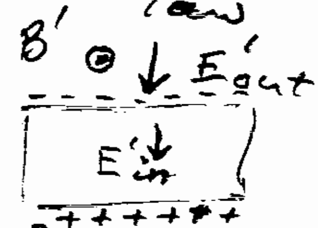
b) Wire frame: $B = 0$, $E = \frac{2\lambda\,\hat{r}}{r}$, $\lambda \equiv \rho n A$ (A = wire crossection)

moving frame: $B' = -\beta\frac{v}{c}\frac{2\lambda}{r}\hat{\varphi}$, $E' = \beta\frac{2\lambda}{r}\hat{r}$

$B'_\varphi = -\beta\frac{v}{c}E_r$, $E'_r = \beta E_r$ — agrees w. field transform law

Problem 2 a)





$$E'_{out} = \beta\left(\frac{v}{c}\times\vec{B}\right)$$

$$E'_{in} = \frac{1}{\varepsilon}E'_{out}$$

$$P' = \frac{1}{4\pi}\left(\frac{1}{\varepsilon}-1\right)E'_{out}$$

$$E_{in} = \beta\left(E'_{in} - \left(\frac{v}{c}\times B'\right)\right) \Leftarrow$$

$$\underline{E_{in} = \beta^2\left(\frac{1}{\varepsilon}-1\right)\left(\frac{v}{c}\times B\right)}$$

$$B = \beta\left(B' + \frac{v}{c}\times E'\right) = \beta\left(1-\frac{v^2}{c^2}\right)B = B \text{ (ok)}$$

$$E_{out} = \beta\left(E' - \frac{v}{c}\times B'\right) = 0 \text{ (ok)}$$

$$\sigma = \beta\sigma' = \beta P'_\perp = \beta^2\frac{1-\varepsilon}{4\pi\varepsilon}\frac{\vec{v}}{c}\times\vec{B}$$

b) Perfectly conducting slab is formally equivalent to $\varepsilon = \infty$, thus

$E_{in} = -\beta^2\frac{v}{c}\times B$ (Faraday's emf.)

$E_{out} = 0$

$\sigma = -\frac{\beta^2}{4\pi}\frac{\vec{v}}{c}\times\vec{B}$

$E'_{in} = 0$

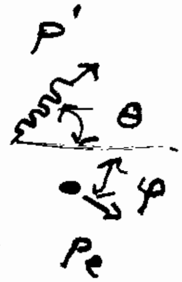
$E'_{out} = \beta\frac{v}{c}\times B$

$P' = -\frac{E_{out}}{4\pi}$

Problem 3

before collision

after collision



a) momentum conserved:

$$p = p'\cos\theta + p_e\cos\varphi$$

$$p'\sin\theta = p_e\sin\varphi$$

Energy conserved:

$$mc^2 + E = E_e + E'$$

$$mc^2 + pc = (p_e^2 + m^2c^2)^{1/2}c + p'c$$

b) Solve, by eliminating φ, p_e

$$(p - p'\cos\theta)^2 + (p'\sin\theta)^2 = p_e^2(\cos^2\varphi + \sin^2\varphi) = (mc + p - p')^2 - m^2c^2$$

$$p^2 + p'^2 - 2pp'\cos\theta = (p - p')^2 + 2mc(p - p')$$

$$pp'(1 - \cos\theta) = mc(p - p')$$

$p = h/\lambda \qquad p' = h/\lambda'$

$$\underline{\lambda - \lambda' = \frac{h}{mc}(1 - \cos\theta)}$$

~~photon~~ wavelength change

Problem 4 a) $\dot{\vec{p}} = e\,\frac{\vec{v}}{c}\times \vec{B}$, $\vec{v} = \frac{c^2}{E}\,\vec{p}$

Take $B \parallel \hat{z}$, then p_z is conserved,

$\frac{dE}{dt} = 0 \rightarrow E$ is conserved $\rightarrow v_z = \frac{c^2}{E} p_z$ is conserved

$$\dot{p}_\perp = \frac{e}{c}\, v_\perp \times B = \frac{ec}{E}\left(p_\perp \times B\right)$$

Solution; $(p_x, p_y) = p^{(0)}\left(\cos(\omega t+\varphi), \sin(\omega t+\varphi)\right)$

$$\underline{\omega = \frac{eBc}{E}}, \quad E = c\left(p_z^2 + p_\perp^2 + c^2 m^2\right)^{1/2}$$

Nonrelativistic limit; $E = mc^2$, $\omega = \frac{eB}{mc}$ (OK)

b) Choose $\vec{E} \parallel x$, $\vec{B} \parallel y$, and go to a moving frame $v \parallel \hat{z}$

$$E_x' = \beta\left(E_x + \frac{v}{c} B_y\right)$$
$$B_y' = \beta\left(B_y + \frac{v}{c} E_x\right)$$

(i) $|E| < |B| \rightarrow E' = 0$ for $\frac{v}{c} = -\frac{E}{B}$

(ii) $|E| > |B| \rightarrow B' = 0$ for $\frac{v}{c} = -\frac{B}{E}$

(i) $E_x' = 0$, $B_y' = \beta^{-1} B_y$

$p_y' = \text{const}$, $(p_z', p_x') = p_\perp^{(0)}\left(\cos\omega(t-t_0), \sin\omega(t-t_0)\right)$ [see part a)]

$\dot{\vec{r}}' = \frac{c^2}{E'}\vec{p}' \rightarrow y'(t) = \frac{c^2}{E'} p_y'(t-t_0) + y_0'$

$$(z', x')(t) = \frac{c^2}{\omega E'}\left(\sin\omega(t-t_0), -\cos\omega(t-t_0)\right) + (z_0', x_0')$$

(ii) $E_x' = \beta^{-1}E_x$, $B' = 0$ $\quad p_{y,z}' = \text{const}$ $\quad p_x' = eE_x'(t-t_0) + p_x^{(0)}$

$\dot{y}' = \frac{c^2}{E'} p_y' \rightarrow y' = \frac{p_y' c}{eE_x'}\,\text{asinh}\,\frac{eE_x'(t-t_0) + p_x^{(0)}}{(p_y^2 + p_z^2 + m^2c^2)^{1/2}}$ $\quad E' = \left(p_x^2(t) + p_y^2 + p_z^2 + m^2c^2\right)^{1/2} c$

z': same as $y'(t)$

$$\dot{x}' = \frac{c^2}{E'} p_x'(t) \rightarrow x'(t) = \frac{c}{eE_x'}\left(\left(eE_x(t-t_0) + p_x^{(0)}\right)^2 + p_y'^2 + p_z'^2 + m^2c^2\right)^{1/2} + x_0'$$

Problem 5 $\quad M = \frac{\partial L}{\partial \dot\varphi} = \frac{m\rho^2\dot\varphi}{\sqrt{1-(\dot\rho^2+\rho^2\dot\varphi^2)/c^2}} \equiv \beta m \rho^2 \dot\varphi$

$$E = \dot\rho\frac{\partial L}{\partial\dot\rho} + \dot\varphi\frac{\partial L}{\partial\dot\varphi} - L = m\frac{\dot\rho^2+\rho^2\dot\varphi^2}{\sqrt{\quad}} + mc^2\sqrt{\quad} - \frac{e^2}{\rho} = \frac{mc^2}{\sqrt{\quad}} - \frac{e^2}{\rho}$$

$$\sqrt{\ldots} = \frac{mc^2}{E+\frac{e^2}{\rho}} \equiv \beta^{-1}$$

a) circular trajectories, $\dot\rho = 0$, $\dot\varphi = \frac{2\pi\rho}{T}$

$$\dot{\mathbf{p}} = -\frac{e^2}{\rho^2}\hat\rho \quad \to \quad \beta m v\dot\varphi = \frac{e^2}{\rho^2} \to \beta m\rho\dot\varphi^2 = \frac{e^2}{\rho^2}$$

$$\frac{T^2}{\beta R^3} = \text{const} \qquad (\text{relativistic } T-R \text{ relation})$$

b) $m\rho^2\dot\varphi = \frac{M}{\beta} = \frac{mc^2}{E+\frac{e^2}{\rho}}M \qquad \dot\varphi = \frac{c^2M}{\rho^2(E+\frac{e^2}{\rho})}$

$$1 - \frac{\dot\rho^2}{c^2} - \frac{\rho^2\dot\varphi^2}{c^2} = \left(\frac{mc^2}{E+\frac{e^2}{\rho}}\right)^2$$

$$\frac{1}{c}\dot\rho = \left(1-\left(\frac{mc^2}{E+\frac{e^2}{\rho}}\right)^2 - \frac{e^2}{c^2}\frac{c^4M^2}{\rho^4(E+\frac{e^2}{\rho})^2}\right)^{1/2} = \left(1 - \frac{(mc^2)^2+(cM/\rho)^2}{(E+e^2/\rho)^2}\right)^{1/2}$$

$$c\frac{d\varphi}{d\rho} = \frac{c^2M/\rho^2}{\left[\left(E+\frac{e^2}{\rho}\right)^2-(mc^2)^2-\left(cM/\rho\right)^2\right]^{1/2}} \qquad u \equiv \frac{1}{\rho}$$

$$-\varphi = cM\int_{u_0}^{u_1}\frac{du}{\left((E+e^2u)^2-(mc^2)^2-(cMu)^2\right)^{1/2}}$$

$u_{0,1}$ - turning points, zeroes of $\sqrt{\ldots}$

$$\Delta\varphi = \frac{cM\pi}{[(cM)^2-e^2]^{1/2}} = \left(1-\frac{e^2}{M^2c^2}\right)^{-1/2}\pi$$

precession at $v/c \ll 1$ by $\delta\varphi = -\frac{2e}{Mc}\pi$ / cycle

Problem 6 a) In rocket frame, after ejecting δm fuel mass

momentum conserve... $(M-\delta m)\,\delta v = \delta m(u-\delta v)$

$$M\,\delta v = u\,\delta m \quad \text{ignore } \delta^2 \text{ terms}$$

$$-\frac{dM}{M} = \frac{\delta v}{u} \Rightarrow \ln\left(\frac{M_0}{M}\right) = \frac{v}{u}$$

b) $M\,\delta V = \dfrac{\delta m\, u}{\sqrt{1-u^2/c^2}}$ Rocket frame

Lab frame

$$\delta v_L = \left(1-\frac{v_L^2}{c^2}\right)\delta v = \left(1-\frac{v_L^2}{c^2}\right)\frac{u}{\sqrt{1-\frac{u^2}{c^2}}}\,\frac{\delta m}{M}$$

Energy conservation: $(M+\delta M)c^2 + \dfrac{\delta m c^2}{(1-u^2/c^2)^{1/2}} = Mc^2$

$$-\frac{\delta m}{(1-u^2/c^2)^{1/2}} = \delta M$$

$$-\delta v_L = (1-v_L^2/c^2)^2\, u\, \delta M/M$$

$$\frac{c}{2}\ln\frac{1+v_L/c}{1-v_L/c} = u \ln\frac{M}{M(v)}$$

$$\left(\frac{1+v/c}{1-v/c}\right)^{c/2u} = \left(\frac{M_0}{M(v)}\right)$$

Note: for $u, v \ll c$

$$\left(\frac{1+v/c}{1-v/c}\right)^{c/2u} \approx e^{v/u}$$