Problem Set #3

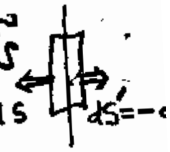
Problem 1 a) consider a thin loop

$\oint \vec{H}\, d\vec{\ell} = \int \left(\frac{1}{c}\frac{\partial D}{\partial t} + \frac{4\pi}{c} j \right) dA \to 0$ as $a \to 0$

$\underline{\vec{H}_1 \cdot d\vec{\ell} = \vec{H}_2 \, d\vec{\ell}}$ Similarly, $\oint E\, d\ell \to 0$ as $a \to 0$ $\underline{\vec{E}_1 \cdot d\vec{\ell} = \vec{E}_2 \cdot d\vec{\ell}}$

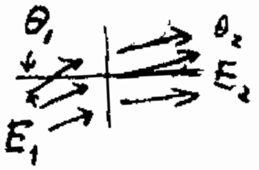
For normal components, use a gaussian pillbox

$\oint \vec{D} \cdot d\vec{S} = 0 \to \underline{\vec{D}_1 d\vec{s} = \vec{D}_2 d\vec{s}}$ similarly, $\underline{\vec{B}_1 d\vec{S} = \vec{B}_2 d\vec{s}}$



Obtain: $B_\perp, D_\perp, H_\parallel, E_\parallel$ continuous

b)



$E_2 \sin\theta_2 = E_1 \sin\theta_1$ ($E_\parallel$ contin)

$\varepsilon_2 E_2 \cos\theta_2 = \varepsilon_1 E_1 \cos\theta_1$ ($D_\perp$ contin)

$\to \underline{\tan\theta_2 = \frac{\varepsilon_2}{\varepsilon_1} \tan\theta_1}$

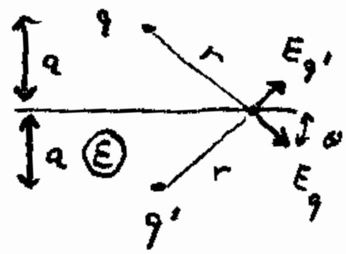
c) Similarly

$H_2 \sin\theta_2 = H_1 \sin\theta_1$

$\mu_2 H_2 \cos\theta_2 = \mu_1 H_1 \cos\theta_1$

$\to \underline{\tan\theta_2 = \frac{\mu_2}{\mu_1} \tan\theta_1}$

Problem 2 a) $E_\parallel$ cont. $\frac{q}{r^2}\cos\theta + \frac{q'}{r^2}\cos\theta = \frac{q''}{r^2}\cos\theta$



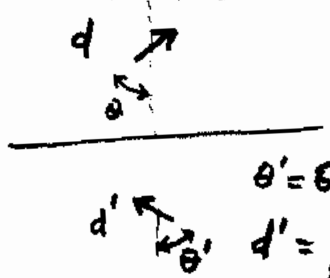
$D_\perp$ cont. $\frac{q}{r^2}\sin\theta - \frac{q'}{r^2}\sin\theta = \varepsilon \frac{q''}{r^2}\sin\theta$

$\begin{cases} q + q' = q'' \\ q - q' = \varepsilon q'' \end{cases}$

$q'' = \frac{2}{\varepsilon+1} q$

$q' = -\frac{\varepsilon-1}{\varepsilon+1} q$ $\quad F_q = \frac{q q'}{(2a)^2} = -\frac{\varepsilon-1}{4(\varepsilon+1)} \frac{q^2}{a^2} \hat{z}$ (attraction)

b) Representing dipole as two +, − charges, and using superposition obtain image dipole:



Interaction energy $U_{12} = \frac{\vec{d}\vec{d}' - 3(\vec{d}\vec{n})(\vec{d}'\vec{n})}{(2a)^3} =$

$= \frac{dd'}{(2a)^3}(\cos 2\theta - 3\cos^2\theta) = -\frac{dd'}{(2a)^3}(1 + \cos^2\theta)$

$d' = \frac{\varepsilon-1}{\varepsilon+1} d$ $\quad$ Force $F = -\frac{\partial U_{12}}{\partial(2a)} = -3\frac{dd'}{(2a)^4}(1+\cos^2\theta)$ (attraction)

c) Solution can be obtained by replacing $E, D, \varepsilon, d \to H, B, \mu, m$ in formulas & drawing of part b)

$m' = \frac{\mu-1}{\mu+1} m$ $\quad$ Force $F = -\frac{\partial U_{12}}{\partial(2a)} = -3\frac{mm'}{(2a)^4}(1+\cos^2\theta)$ (attraction)

$\theta' = \theta$

Problem 3 $\left(\frac{\partial^2}{\partial t^2} - \frac{1}{v^2}\frac{\partial^2}{\partial x^2}\right) f = 0$

a) using $\xi, \eta = x \pm vt$, have $\frac{\partial^2}{\partial \xi \partial \eta} f = 0 \to \underline{f = f_L(\xi) + f_R(\eta)}$

b) $F_1(x) = f_{t=0} = f_{L\,t=0} + f_{R\,t=0}$, $F_2(x) = \partial_t f = v\partial_x f_{L(t=0)}(x) - v\partial_x f_{R(t=0)}$

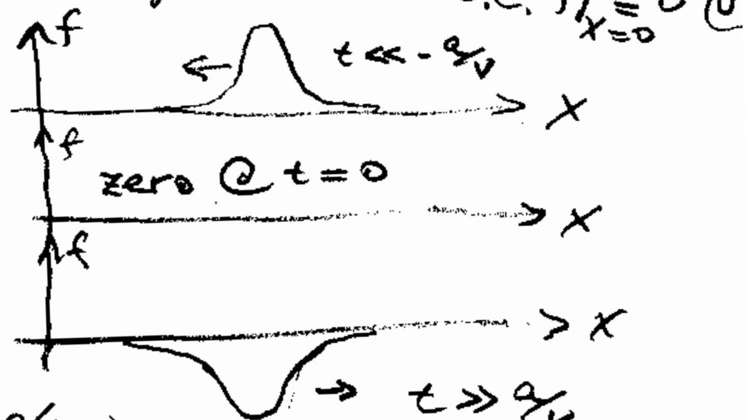
$t=0$: $f_{L,R}(x) = \frac{1}{2}\left(F_1(x) \pm \frac{1}{v}\int^x F_2(x')dx'\right)$ $\quad \int^x F_2(x')dx' = v f_L(x)_{t=0} - v f_R(x)_{t=0}$

$$\underline{f(x,t) = \frac{1}{2}\left(F_1(x-vt) + F_2(x-vt)\right) + \frac{1}{2v}\int_{x-vt}^{x+vt} F_2(x')dx'}$$

c) Choose $f_R(z) = -f_L(-z)$, then $f(x,t) = f_L(x+vt) - f_L(-x+v$

an "image wavepacket" b.c. $f|_{x=0} \equiv 0$ @

For $f_L = e^{-a(x+vt)^2}$,
have $f_R = -e^{-a(x-vt)^2}$



d) Try $f_R(z) = r f_L(-z)$
reflection coefficient

B.C. @ $x=0$:

$v f_L'(vt) + r v f_L'(vt) = u f_L'(vt) - r u f_L'(vt)$

$\underline{r = \frac{u-v}{u+v}}$ 1) no refleted wave for $u=v$ (impedance matching condition)

2) $-1 < r < 0$ for $0 < u < v$ 3) $0 < r < 1$ for $u > v$

Problem 4

$U = \frac{E^2}{8\pi} + \frac{B^2}{8\pi} = (E_0^2 + B_0^2)\frac{f^2}{8\pi}$ $\quad G/U = \frac{2E_0 B_0}{c(E_0^2 + B_0^2)}$

$\vec{G} = \frac{1}{4\pi c} E \times B = E_0 B_0 \frac{f^2}{4\pi c}\hat{z}$

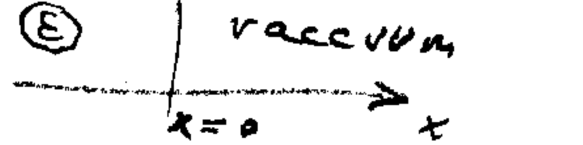
separate into left and right packets

$E_0 = E_R + E_L$
$B_0 = B_R + B_L$
$B_R = E_R$, $B_L = -E_L$

$\nearrow$ $E_R = \frac{1}{2}(E_0 + B_0)$ $\quad \frac{U_{R,L}}{U} = \frac{(E_0 \pm B_0)^2}{2(E_0^2 + B_0^2)}$

$E_L = \frac{1}{2}(E_0 - B_0)$ $\quad$ One packet for $\alpha = \frac{E_0}{B_0} = \pm 1$

Problem 5



a) E, B obey $\left(\frac{\partial^2}{\partial x^2} - v^2 \frac{\partial^2}{\partial t^2}\right) f = 0$ with $v = c$ in vacuum and $v = \frac{c}{\sqrt{\varepsilon}}$ in dielectic

Boundary conditions: $B_{\parallel}$ $E_{\parallel}$ continuous

b) $\underset{x=-0}{B_d} = \underset{x=+0}{B_v} \rightarrow \frac{\partial B_d}{\partial t} = \frac{\partial B_v}{\partial t} \xrightarrow{\text{Faraday law}} \left.\frac{\partial E_d}{\partial x}\right|_{x=-0} = \left.\frac{\partial E_v}{\partial x}\right|_{x=+0}$

$\underset{x=-0}{E_d} = \underset{x=+0}{E_v} \rightarrow \frac{\partial E_d}{\partial t} = \frac{\partial E_v}{\partial t}$

For a wave incident from vacuum, the wave in dielectric is propagating to the left

$E_d(x,t) \sim f(x+vt)$, thus $\frac{\partial E_d}{\partial t} = \frac{c}{\sqrt{\varepsilon}} \frac{\partial E_d}{\partial x}$

Eliminate E_d from boundary conds.:

$$\frac{\partial E_v}{\partial t} = \frac{\partial E_d}{\partial t} = \frac{c}{\sqrt{\varepsilon}} \frac{\partial E_d}{\partial x} = \frac{c}{\sqrt{\varepsilon}} \frac{\partial E_v}{\partial x}$$

This b.c. has the same form as b.c. in Prob 4 d) reflection coefficient is

c) $r = E_r / E_0 = \frac{u-v}{u+v} = \frac{c/\sqrt{\varepsilon} - c}{c/\sqrt{\varepsilon} + c} = -\frac{\sqrt{\varepsilon}-1}{\sqrt{\varepsilon}+1}$

$E_t = E_d|_{x=0} = E_v|_{x=0} = E_0 - \frac{\sqrt{\varepsilon}-1}{\sqrt{\varepsilon}+1} E_0 = \frac{2}{\sqrt{\varepsilon}+1} E_0$

$t = E_t / E_0 = \frac{2}{\sqrt{\varepsilon}+1}$

For the wave incident from dielectric side,

$v = c/\sqrt{\varepsilon}$, $u = c$, $r = \frac{E_r}{E_0} = \frac{u-v}{u+v} = \frac{\sqrt{\varepsilon}-1}{\sqrt{\varepsilon}+1}$, $t = 1+r = \frac{2\sqrt{\varepsilon}}{\sqrt{\varepsilon}+1}$