8.311: Electromagnetic Theory Problem Set # 1 Due: **2/10/04**

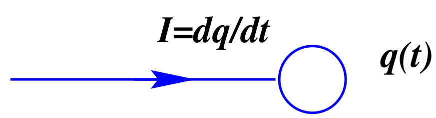
Conservation Laws in Electromagnetism

Reading: Schwinger, Chap. 1,3 (or Jackson, Chap. 6)

1. Displacement current.

a) Consider the displacement current $\mathbf{j}_d = \frac{1}{4\pi}\partial\mathbf{E}/\partial t$ of an electromagnetic field, and show that the sum of the actual electric current $\mathbf{j}$ and $\mathbf{j}_d$ is divergenceless: $\nabla \cdot \mathbf{J} \equiv \partial_i J^i = 0$, $\mathbf{J} = \mathbf{j} + \mathbf{j}_d$.

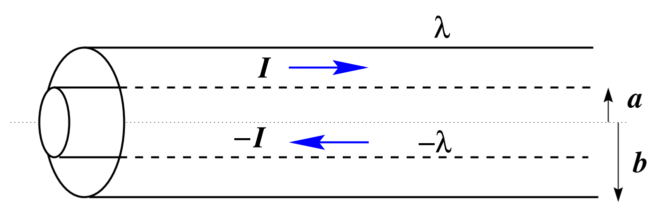
b) A conducting sphere is being charged through a straight thin wire carrying current I, so that the charge on the sphere varies in time as $\dot{q} = I$. Assuming a symmetric distribution of charge over the sphere surface, find the electric field outside the sphere. Determine the displacement current, and verify the conservation law $\nabla \cdot \mathbf{J} = 0$.



c) Using Ampère-Maxwell law in the form $\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}$, find the magnetic field. Verify that close enough to the wire the answer has the familiar Ampère's straight infinite wire form.

2. Poynting vector.

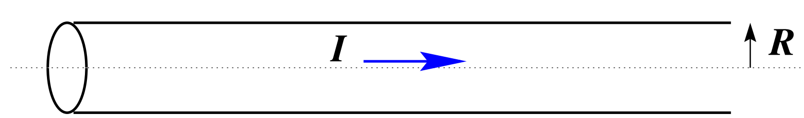
a) A circular cylindrical transmission line is made of two coaxial cylinders with the radii a and b, $a < b$. The outer cylinder carries current I, that returns along the inner cylinder, and the cylinders' charge per unit length is λ and $-\lambda$, respectively. Find the electric and magnetic field, and determine the energy flux $\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}$. By integrating $\mathbf{S}$ over the cross-section, obtain the total energy flux in the transmission line By a calculation show that, if a resistor R is connected to an end of the line, the power dissipated equals the total energy flow due to $\mathbf{S}$.



b) For the system described in Problem 1, consider Poynting vector distribution in space. Make a sketch, showing the energy flow. Discuss the energy balance.

3. Poynting vector in a conductor.

Consider a long cylindrical wire of radius R carrying a steady current I. The conductivity of the material is σ. The magnetic field of the current is transverse to the electric field, which gives rise to the energy flux. Find the Poynting vector outside and inside the wire. Calculate the flux of the Poynting vector through a cylinder of radius r coaxial with the wire, compare with the Joule heat produced by current, and verify the energy conservation in this system.



4. Maxwell electric stress tensor

Using the stress tensor of the electric field

$$T_{ij} = \frac{1}{4\pi}\left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right),$$

one can consider forces in electrostatics. As the following two examples illustrate, this approach becomes especially convenient for problems with complicated geometry and/or distributed charge density.

a) Show that the pressure of electric field on a surface of a charged conductor is equal to $\mathbf{E}^2/8\pi$, where $\mathbf{E}$ is the field near the surface. Obtain the same answer by using the formula $\mathbf{F} = \sum_i q_i \mathbf{E}(\mathbf{r}_i)$ (be careful with the factors of two).

b) Consider a ball of radius R charged uniformly throughout its volume with a total charge Q. By using the stress tensor, find the force that the lower hemisphere exerts on the upper hemisphere. Show that you get the same result by evaluating the net Coulomb force.

5. Maxwell magnetic stress tensor

A long cylindrical solenoid of radius r with N turns of wire per unit length carries constant current I.

a) Using the stress tensor of the magnetic field,

$$T_{ij} = \frac{1}{4\pi}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right),$$

find the pressure of the field on the solenoid surface.

b) Calculate the pressure by using the Lorentz force on a wire, $d\mathbf{F} = \frac{1}{c} I \, d\mathbf{l} \times \mathbf{B}$. (Again, be careful with the factors of two)