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PROFESSOR: I'd like to begin building up momentum by going over what we've already done to kind of see how it all fits together. So I put the last lecture summarised on transparencies here. And we'll go through them quickly, and then we'll start new material, which I'll be doing on the blackboard.

So we've been studying this mathematical model of a universe obeying Newton's laws of gravity. We considered simply a uniform distribution of mass, initially spherical, and uniformly expanding, which means expanding according to Hubble's law, with velocities proportional to the distance from the origin. And Newton's laws then tell us how that's going to evolve. And our job was just to execute Newton's laws to calculate how it would evolve.

And we did that by describing the evolution in terms of a function little r , which is a function of r , i , and t . And that's the radius at time t of the shell that was initially a radius r sub i . So we're trying to track every particle in this sphere, not just the particles on the surface. And we want to verify that it will remain uniform in time. Matter will not collect near the edge of the center. And we discovered it will remain uniform if we have a 1 over r squared force law, but for any other force law it will in fact not remain uniform.

So we derived the equations and found that it obeyed this scaling relationship, which is what indicated the maintenance of uniformity, wholesale scale by the same factor, which we then called a of t . So the physical distance of any shell from the origin at any time is just equal to a of t times the initial distance from the origin.

And furthermore, we were able to drive equations of motion for the scale factor. And it obeys the equations, which in fact were derived from general relativity by

Alexander Friedmann in 1922, and therefore they're called the Friedmann equations. There's a second order equation, a double dot, which just tells us how the expansion slowed down by Newtonian gravity. a double dot is negative, so the expansion is being slowed by the gravitational attraction of every particle in this spherical distribution towards every other particle.

And we also were able to find a first order equation by integrating the second order equation. And the first equation has this form. It could be written a number of different ways depending on how you arrange it. But this is the way that I consider most common. And many books refer to the second equation as the Friedmann equation. Both of these equations were derived by Alexander Friedmann. I think it's perfectly called both Friedmann equations. But most books do not do that.

And in addition to finding the equations of evolution for $a(t)$, we also understood how $\rho(t)$ evolves. And that really is pretty trivial to begin with. The Newtonian mass of this sphere stays the same. It just spreads out over a larger volume as $a(t)$ grows. So if the mass stays the same and the volume grows as a^3 , then the density has to go like one over a^3 . And this alone could be written more precisely as an equation by saying the easiest way to view the logic is that this equation implies that $a^3 \rho$ is independent of time. And then once you know that $a^3 \rho$ is independent of time, you can write the equation in this form, which if I multiply a^3 to the left hand side of the equation, we can just say that $a^3 \rho$ is the same at time t as it is at some other time t_1 .

Now in lecture last time, we wrote this equation where t_1 was t_i the initial time, and $a(t_i)$ was one. So we didn't include that factor. So this is slightly more general way of writing it than we did last time. But it still has no more content than the statement that $\rho(t)$ falls as $1/a^3(t)$.

We introduced a special set of units to describe this. Question, yes?

AUDIENCE:

I had a question about the Friedmann equations and what we're calling e^2 , and the interpretation of that.

PROFESSOR: What we're calling what squared?

AUDIENCE: h squared.

PROFESSOR: h squared, yes.

AUDIENCE: So in the spherical universe, we showed that if e was positive that corresponded to an open universe that would expand forever. So doing the p set for this week for the cylindrical universe, I think we found that that universe would collapse.

PROFESSOR: Yes.

AUDIENCE: E was also positive for that one. So I was just wondering about the interpretation of [INAUDIBLE]

PROFESSOR: Right, well I'll be coming to that issue later. Let me come back to that, OK. Since I haven't introduced e yet. And the slide is here. So an interesting question. We'll come back to that.

So to describe this system of equations, I like to introduce this notion of a notch, which is a special unit used only to measure co-moving coordinates. And in this case, $r_{sub\ i}$ is our co-moving coordinate. As our shells move, we label them all by where they were a time t_i . We don't change those labels. So those are the co-moving coordinates, a coordinate system that expands with the universe.

So instead of measuring r_i in meters or any other physical length, I like to measure them in a new unit called a notch, just to keep things separate. And a notch is defined so that a of t is measured in meters per notch, and at time t_i , one notch equals 1 meter. But at different times, the relationship between notches and meters is different because it's given by this time dependent scale factor of t .

If [INAUDIBLE] works to have things depend on these units, one find that this new quantity that we introduced, little k , which all we know is that it's a constant, has the units of 1 over a notch squared. And that has some relevance to us, because it means that we can make little k have any value we want by choosing different

definitions for the notch. And the notch is up for grabs. We are just inventing a unit to use to measure our co-moving coordinate system.

So we can always adjust the meaning of a notch so that k has whatever value we want. As long as we can't change its sign by changing the units. And if it's zero we can't change it by changing its units. As long as non-zero we can make it any value we want, and in fact that is often used in many textbooks.

And that's I guess what I want to talk about next, the conventions that are used to define a of t . And for us, I'm going to treat this notch as being arbitrary. We've defined the notch originally so that a of t_i was one meters per notch at time t_i , and that gave the notch more or less a specific meaning. But the specific meaning depends on what t_i is. One can take the same relationship and view it as simply a definition of t_i . t_i is the time at which the scale factor is one meter per notch. We take the definition of t_i and we can let the notch be anything we want, and there will be some time t_i that will still make that statement true.

So what I want to do basically is to think of this equation, a of t_i equals 1 meter per notch, not as a definition of a notch, which I want to leave arbitrary, but rather as a definition of t_i . And after defining t_i , I want to just forget about t_i . t_i in fact will not enter our equations anywhere. So we don't need to remember its definition.

I decided like the cubit. All of us know that the cubit is some unit distance, but we don't care what it is because we never use cubits. Same thing here. We'll just never use t_i . And since we're never going to use it. We don't need to remember how it was initially defined. It's only of historical interest.

So the bottom line then is simply that for us the notch is just an undefined unit of distance in the co-moving coordinate system. Other people use different definitions. Ryden, for example uses the definition where a of t sub, a of the present time is equal to 1. And we would interpret that as meaning one meter per notch today. And that's a perfectly good definition. And we can use it whenever we want, because our notch is initially undefined. So that allows us the freedom to define it in any

particular problem in any way that we want.

Many other books take advantage of the fact that this quantity k has units of inverse notch squared, even though they don't say that. But that means you could rescale the co-moving coordinate system to make k equal to whatever value you want. So in many books k is always equal to plus or minus 1 if it's non-zero. The co-moving coordinate systems is just scaled. We would do is we scaling of the notch to make k have magnitude 1.

OK, having derived these equations, the next step is to go about asking what do the solutions to the equations look like. And that's where things start getting interestingly when we start getting some real nontrivial results. The Freedman equation, the first order one, could be rewritten this way. It's just a rewriting of rearranging things. And I used here the fact that ρ times a^3 was a constant. If we took our original form of the Freedman equation, this would be $\rho_{sub i} \text{ times } a^3 \text{ of } t_{sub i}$, which would be 1.

But knowing that ρ times a^3 is a constant, we can let t argument here be anything we want, which is what I'll do, just to emphasize that we don't care anymore what $t_{sub i}$ was. It really has disappeared from our problem. It was just our way of getting started.

So this equation holds. And we can use it to discuss how the different classes of solutions will behave. And what we can see very quickly from this equation is that the behavior of the solutions will depend crucially on the sign of k . And it's useful here to remember, although we don't need to know anything more than this proportionality, that k is proportional to the negative of something that we call e . And that the thing that we call e is related to the overall energy of this thing.

So we'll keep that in the back of our minds. But it's really only for intuition. Everything that we're going to say follows directly from this equation, where we don't know anything about e . There are three types of solutions, depending on whether k is positive, negative, or zero, and those are the options for what k might be. It's a real number.

So first we consider with the solutions where k is less than zero, which means e is greater than zero. And e being greater than zero means the system has more energy than zero. And in this case, zero energy would correspond to having all the particles infinitely far away. So the potential energies would be zero. And all particles the rest, so the kinetic energies would be zero.

So in particular, zero energy would correspond to the system being completely dispersed, no longer compact. And in this case, our system has more energy than that. And more energy than that means it can blow outward without limit. And we see that directly from the equation.

If the second term has a negative value of k , then the second term itself is positive. And the first term is also always positive. And that means that a dot squared, no matter what happens to the first term, is always at least bigger than the second term, which is a constant. And if a dot is always bigger than some constant means that a grows indefinitely. And that's called an open universe. And it goes on expanding forever.

Second case we'll consider is k greater than zero, which corresponds to e less than zero. And that means since 0 corresponds to the system being completely dispersed, e less than 0 means the system does not have enough energy to ever become completely dispersed. So we'll have some maximum size. And the maximum size follows immediately from this equation. a dot squared has to be positive. It can ever get negative. It can become zero, but it can never get negative.

In this case, the minus $k c$ squared term is negative. And that means that if this term gets to be too small, the sum will be negative, which is not possible. So this term cannot get to be too small. And since a of t is in the denominator, that means a of t cannot get to be too big. And you can easily derive the inequality that a has to obey for the right hand side to always be positive. And in that case you [INAUDIBLE] a max, which is what you would get if you just set the right hand side equal to 0, given by that expression. a can never get bigger than that, because if it did, the right hand side of the equation become negative, which is not possible.

So this universe will reach a maximum size, which we just calculated, and then [INAUDIBLE] will come back. So we already have a very nontrivial result here. Given a description of a universe of this type, we can calculate how big it will get before it turns around and collapses. And this of universe ultimately undergoes a big crunch when it collapses, where the word big crunch was made as an analogy to the phrase big bang. It's called a closed universe.

And then finally, we've now considered the k less than zero and k greater than zero. There are many cases when k equals zero. And that's called the critical mass density or critical universe. And we can figure out what it means in terms of the mass density. This again is our Friedmann equation. If k is zero, this last term is absent. So we just have a relationship between ρ and h , the Hubble expansion rate.

And we can solve that. And the value of ρ which satisfies that equation is called the critical density. As the density is equal to the critical density, it means that k is zero. And that's called a flat case. We'll figure out in a minute how it evolves. It's not clear if it will be collapsed or stop or what. But we'll find out soon. It's really on the borderline between something which we know expands and something which we know collapses.

And it's called a flat universe. The word flat suggests geometry, and we'll be learning about that later. General relativity tells us a little bit more than we learn here. These equations are all exactly true in the context of general relativity. But general relativity also tells us that these equations are connected to the geometry of space. And only for this critical mass density is the space Euclidean. The word flat here is used in the sense of Euclidean.

So to summarize what we've said, if the mass density is bigger than this critical value, we get a closed universe, which reaches a maximum size and then collapses. If the mass density is less than the critical density, we get an open universe, which goes on expanding forever. And if the mass density is exactly equal to the critical density, that's called a flat k . So we'll explore a little bit more in a minute.

It's interesting to know what this critical density is. It depends on the expansion rate. But the expansion rate has now been measured quite accurately. So if I take the value of 67.3, which is the value that comes from the Planck satellite combining their results with several other experiments, they get a value of 67.3. And when we'll put that into this formula, the number one gets is 8.4×10^{-30} grams per centimeter cubed, which is only about 5 proton masses per cubic meter. It's an unbelievably empty universe that we live in.

I say the universe that we live in because in fact the mass density of our universe is very close to this critical value. It's equal to it to within about half of a percent we now know.

An important definition, which we'll be continuing to use through the course and which cosmologists always use, is Ω , where Ω means capital Greek Ω . And that's just defined to be the actual density of the universe, whatever it is, divided by this critical density.

OK, the one remaining thing that we did last time, and we'll summarize this and go on, is we figured out what the evolution is for a flat universe. And we can do that just by solving the differential equation, which is a fairly simple differential equation. If we leave out the k term, the Friedmann equation becomes \dot{a}^2 is equal to $8\pi G/3 \times \rho a^2$. And we know how ρ depends on a . It's proportional to $1/a^3$. So the right hand side here is some constant divided by a^3 .

And by just rearranging things, we can rewrite that as da/dt is equal to some constant over $a^{1/2}$. In this slide I use this symbol const a number of times. Those constants are not all equal to each other. But they're all constants, which you can keep track of if you wanted to. But there's no need to keep track of them because they have no bearing on the answer anyway. So I just called these constants const .

So again, da/dt equals $\text{const}/a^{1/2}$, which is an easy

differential equation to solve. We just multiply through by dt and a to the one half, and write it in as a to the one half times da equals constant times dt , which can easily be integrated both sides of the equation as indefinite integrals. And then we get $\frac{2}{3}$ times a to the $\frac{3}{2}$ is equal to a constant times t , where this constant happens to be the same as that constant, for whatever that's worth, plus a new constant of integration, c prime.

Then we argue that the value of c prime depends on how we synchronize our clocks. If we reset our clock by changing t by a constant that would change the value of c prime. And since we haven't said anything yet about how we're going to set our clock, we're perfectly free at this point to just say that we're going to set our clock so that t equals 0 corresponds to the same time that a is equal to zero. The initial singularity of the universe starts from zero size with a as zero.

So if we do that when a is zero, t is zero. That means that c prime is zero. So setting c prime equal to zero is just a choice of how to set our clocks. So we do that. And then we can take the $\frac{2}{3}$ power of this equation. And since constants are just constants that we don't care about, we end up with a of t is proportional to t to the $\frac{2}{3}$.

And proportional is all we need to know, because the constant of proportionality would depend on the definition of the notch. And we haven't defined the notch. And we don't need to define the notch. And so anything that depends on the constant of proportionality will never enter any physical answer. It will be relevant to questions like how many notches are a certain distance on your map. But for any physical answer, we don't care. If we want to talk about our map, we could just define a notch to be whatever we want it to be.

OK, that's the end of my summary. Any questions about any of that? I'm sorry. I didn't come back to answer your question about the cylinder. The cylinder problem does end up always giving you a closed universe no matter what the parameters are. It always collapses. And even though its energy as you compute it would turn out to be positive, the difference though is that for the case of the cylinder, the

potential energy does not go to 0 as the thing becomes infinitely big. The potential energy has a logarithmic diversion in it. So the zero is just placed differently for the case of the cylinder problem. So it ends up being closed no matter how fast it's expanding. Yes?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Well that is true. Certainly the differential equations break down when a is equal to zero. The mass density goes to infinity. But we're still certainly free to set our clocks so that the equations themselves, when extrapolated to 0, would have the property that a equals zero when t is equal to zero. It's certainly correct, and I was going to be talking about this in a minute, that you should not trust these equations back to t equals zero. But that doesn't stop you for choosing whatever you want as your origin of t . And if these are the equations we have, the simplest way to deal with these equations is to use the zero of t when equations say that a was zero.

Any other questions? OK, in that case, we will leave the slides for a bit and proceed on the blackboard. So so far I think we have learned two varied nontrivial things from this calculation. We learned how to calculate the critical density. We learned how to calculate what density the universe would have to have so it would re-collapse. And we've also learned that if the universe is closed, we can calculate how large it will get before it collapses. So those are two very nontrivial results coming out of this Newtonian calculation.

The next question I want to ask is still about the flat universe. It's a fairly trivial extension of what we have. Given this formula for a of t , I would like to calculate the age of a flat universe. If you were living in a flat universe that was matter dominated like the one we're describing, how would you determine how old it was?

And the answer is that it's immediately related to the Hubble expansion rate, and the age can be expressed in terms of the Hubble expansion rate. To see that-- so we're calculating the age of a matter dominated flat universe. And these age calculations will be extending as we go through the course. So in the end, we'll have the full calculation for the real model that we have of our universe.

But you have to start somewhere. So we're starting with just a flat, matter dominated universe. We know that \dot{a} of t is equal to some constant, which I will call little v times t to the $2/3$ power. Previously I just used proportional to, but now it's just more convenient to give a name to the constant of proportionality. We'll never need to know what it is. But v is some constant of proportionality.

This by the way already tells us something else, which wasn't obvious from the beginning, which is our flat universe does go on expanding forever, somewhat like an open universe. An important difference is that if you calculate $\frac{da}{dt}$ for the open universe, that approaches a constant as time goes to infinity. That is, the universe keeps on expanding at some-- minimal rate forever. In this case, if you calculate the $\frac{adt}{a^2}$, it goes to 0 at times. So the flat universe expands forever, but at an ever, ever, ever decreasing rate. We know how to relate \dot{a} of t to h . The Hubble expansion rate is $\frac{\dot{a}}{a}$, we learned a long time ago.

And if we know what \dot{a} is, we know what this is. So this is just 2 over $3t$. The $2/3$ coming from differentiating the $2/3$. And that gives you \dot{a} of t to the minus $1/3$, but then you're also dividing by a , which turns that t to the minus $1/3$ to a t to the minus 1 . So you get 2 over $3t$ is the final answer.

So this is the relationship now between h and t , and the question we asked is how to calculate the age. The age is t . This is all defined as where t is equal to 0 at the big bang. So t really is the time elapsed since the big bang. So we're left immediately with a simple result, that t is equal to $2/3$ times h inverse.

Now this result immediately makes rigorous contact with something that we talked about in vague terms earlier. If you are so unfortunate as to badly mis-measure h , you can get a pretty wild answer for the h of your model universe. And Hubble mis-measured h by about a factor of seven comparative to present modern values. He got h to be too high. His value of h was too high by a factor of about seven, and that meant that when big bang theorists calculated the age of the universe were getting ages that were too low by a factor of seven. And in particular that meant they were getting ages of the order of 2 billion years. And even back in the 1920s and '30s,

there was sufficient geological evidence that the Earth was older than 2 billion years. There was also significant understanding of stellar evolution, that starts took longer to evolve than 2 billion years. So the big bang model was in trouble from the start, largely because of this very serious mis-measurement in the early days of the Hubble expansion rate.

If we put in some numbers, this of course is not an accurate model for our universe, we now know. Our universe is now currently dark energy dominated. But nonetheless, just to see how this works, we can put in numbers. So h , using this Planck value that I quoted earlier 67.3 plus or minus 1.2 kilometers per second per megaparsec. To be able to get an answer in years, one has to be able to convert this into inverse years. h is actually an inverse time. And a useful conversion number is 1 over 10^{10} years is equal to almost 100, but not quite, 97.8 kilometers per second per megaparsec, which allows you to convert these Hubble expansion rate units into inverse years. And using that one finds that the age of the universe using the $2/3 h$ inverse formula is 9.7 plus or minus 0.2 billion years.

Now this number played a role in the fairly recent history of cosmology. Before 1998, when the dark energy was discovered, which kind of settled all these questions, but before 1998, we thought the universe what matter dominated. It might have been open. It didn't have to be flat. That was debated. It looked more open and flat.

But some of us wanted to hold out for a flat universe because we were fans of inflation and admired inflation's other successes, which we'll learn about later in the course, and thought therefore that the universe should be flat, and wanted to try to reconcile all this. And the problem was that like cosmology in the '20s and '30s when the age of the universe that you calculate was too young, the same thing was happening here before 1998, when we thought the universe was matter dominated. This is the age that we got, modified a little bit by having different values of h , but pretty close to this.

At the same time, there were calculations about how old the universe had to be to

accommodate the oldest stars. And in the lecture notes I quote a particular paper by Krauss and Chaboyer. Lawrence Krauss is an MIT PhD by the way. And what they decided by studying globular clusters, which are supposed to contain the oldest stars that astronomers know about, that the oldest stars, they said, had an age of 12.6 plus 3.4 minus 2.2 billion years. And this is a 95% confidence number.

That is, instead of using one sigma, which are often used to quote errors, these are two sigma errors, which have probabilities of being wrong by only 5% if things work properly according to the statistics. So then we're going to think about 95% limits. So they were willing to take the lower limit here, which was 10.4. So they got a minimum age of 10.4 billion years for the oldest stars. But they also argued that the stars really could not possibly start to form until about 0.8 billion years into the history of the universe.

And doing a little bit of simple arithmetic there, they decided that the minimum possible age for the universe at the 95% confidence level, would be 10.4 plus 0.8 or 11.2. And 11.2 is older than 9.7, and by a fair number of standard deviations, although [INAUDIBLE] were somewhat bigger in 1998 than they are now.

But in any case, this led to I think what people at the time regarded as a tension between the age of the universe question and the possibility of having a flat universe. A flat universe seemed to produce ages that were too young to be consistent with what we knew about stars. Yet there was still evidence in terms of the desire to make inflationary models work to indicate that the universe was flat. And actually it's also true that by 1998, there was evidence from the Kobe satellite measuring fluctuation in the cosmic background radiation, which also suggested that omega was one, that the universe was flat. So things didn't fit together very well before 1998. And this was at the crux of the argument.

It all got settled with the discovery of the dark energy, which we'll learn how to account for in a few weeks. When we include dark energy in these calculations, the ages go up, and everything does come into accord with the idea now that the age of the universe is estimated at 13.8 billion years. And that's consistent with the Hubble

expansion rate given here, as long as one has dark energy and not just relativistic matter. OK any question about that?

OK next thing I wanted to say a little bit about is what exactly we mean here by age. And the question of what we mean by age does of course connect to the question of how do we think it actually began. Because age means time since the beginning presumably. And the answer really is that we don't know how the universe began.

The big bang is often said to be the beginning of the universe. But I would argue that we don't know that, and I think most cosmologists would agree that we don't know that. As we extrapolate backwards, we're using our knowledge of physics that we measure in laboratories and physics that we confirm with other astrophysical type observations, but nonetheless, as we get closer and closer to t equals zero, the mass density in this approximation grows like 1 over the scale factor cubed, which means it goes up arbitrarily large.

Later we'll learn that when the universe gets to be very young, we have to include radiation and not just relativistic matter. The dark energy is actually totally unimportant when we go backwards in time. It becomes important when we go forwards in time.

But when we put in radiation, it does not solve this problem. The universe still requires a mass density that goes to infinity as we approach t equals zero. People had wondered whether maybe that's an idealization associated with our approximation of exact homogeneity and isotropy which after all do break down at some level. Maybe if we put in a slightly inhomogeneous and slightly anisotropic universe and ran it backwards, maybe the mass density would not climb to infinity. Hawking proved that that was not a way out. The universe would become singular. He didn't really prove the mass density went to infinity, but he proved it becomes singular in other ways as t went to zero no matter what geometry you put in.

So the bottom line is that classical general relativity does predict a singularity of some sort as we extrapolate backwards in time. But the important qualification is that once the mass density goes far above any mass densities that we've had any

experience with, we really don't know how things are going to behave. And we don't really know how classical general relativity holds in that regime. And in fact, we have very strong ways is to believe that classical general relativity will not hold in that regime. Because classical general relativity is after all a classical theory, a theory in which one talks about fields that have definite values at definite times without incorporating the ideas of the uncertainty principle of quantum theory.

So nobody in fact knows how to build a theory in which matter is quantized and gravity is not quantized. So all the smart money bets on the fact that gravity is really a quantum theory, even though we don't yet quite understand it as a quantum theory. And that as we go back in time, the quantum effects become more and more important. So there's no reason to trust classical general relativity as we approach t equals zero, and therefore no reason to really take this singularity seriously.

Furthermore, we'll even see at the end of the course that most inflationary scenarios imply that what we call the big bang is not a unique beginning of the universe. But rather it now seems pretty likely, although we sure don't know, that our universe is part of a multiverse, where we are just one universe in the multiverse, and that the big bang, what we call the big bang, is really our big bang, the beginning of our pocket universe. But before that the space of time already existed. The big bang is just a nucleation of a phase transition. It's not really a beginning. And that there was other stuff that existed before what we call the big bang.

I should add though that the inflationary scenario does not provide any answer whatever to the question of how did it all ultimately begin. That's still very much an open question. And it's clear that inflation by itself does not even offer an answer to that question.

So when we talk about the age of the universe, what are we talking about? What we're talking about is the age, the amount of time that has elapsed, since this event that we call the big bang. The big bang might not have been the beginning of

everything, but certainly the evidence is overwhelmingly strong that it happened. And we could talk about how much time has elapsed since it happened. And that's the t that we're trying to calculate here. And it will be offset by a tiny amount by changing the history in the very, very early stages, but only by a tiny fraction of a second. So the uncertainties of quantum gravity are not important in calculating the age of the universe. Although they are important in interpreting what you mean by it. I think we don't really mean the origin of space and time, but rather simply the time has elapsed since the event called the big bang.

OK any questions about that? All right, next event I want to talk about is that if the universe as we know it began some 13.8 billion years ago to use the actual current number, that would mean that light could only have traveled some finite distance since the beginning of the universe as we know it, meaning the universe since the big bang, and that would mean there would be some maximum distance that we could see things. And beyond that there might be more things, but they'd be things for which the light has not yet had time to reach us.

So that's an important concept in cosmology, the maximum distance that you could see. It goes by the name the horizon distance. If you're sailing on the ocean, the horizon is the furthest thing you can see. So what we want to do now is to calculate this horizon distance in the model that we now understand, the flat matter dominated universe.

And this of course is also a calculation that we will be generalizing as we go through the course learning how to treat more and more complicated cases and more and more realistic cases. So this horizon distance, I should define it more exactly. It's the present distance, and maybe I should even stick the word proper here. I've been usually using the word physical distance to refer to the distance to an object as it would be measured by rulers, which are each along the way moving with the velocity of the average matter at those locations. That is also called the proper distance, which is [INAUDIBLE] calls it.

And this horizon distance is defined as the present proper distance of the most

distant objects that can be seen, limited only by the speed of light. So we pretend we have telescopes that are incredibly powerful and could see anything, any light that could have reached us. But we know the light has a finite propagation time, so take that into account in talking about this horizon distance.

OK so what is the horizon distance going to be? Well remember that the coordinate velocity of light is equal to c divided by a of t . I should maybe start by saying before we get down to details, that you might think naively that the answer should be the speed of light times the age of the universe. That's how far light can travel. And so if the universe was static and just appeared a certain time in the past, that would be the right answer. I would just start off at the beginning and travel at speed c .

But it's more complicated because the universe has been expanding all along. And it started out with a scale factor of 0. And furthermore, what we're looking for is present distance in these objects, and the objects of course have continued to move after the light that we're now seeing has left them. So it's a little more complicated than just c times the speed of light. And we'll see what it is by tracking it through very carefully.

We'll imagine a light beam that leaves from some distant object. And the light beam will get the furthest if it leaves earliest. So we want the earliest possible light beam that could have left this distant object. And that would be a light beam that left at literally t equals zero. So the light beam leaves the object at t equals zero and reaches us today. And we want to know how far away is that object? That's the furthest object that we could see, objects for which we can only see the light that was emitted from the object at t equals zero.

So we're going to use our co-moving coordinate system to trace things. All calculations are done most straightforwardly in the co-moving coordinate system. And we know that light travels in the co-moving coordinate system at the rate of dx/dt is equal to c divided by a of t . And this really just says that as the light passes any observer in this co-moving coordinate system, the observer sees speed c , as special relativity tells us he must. But we need to convert it into notches per second

to be able to trace it through the co-moving coordinate system. And the relationship between notches and meters is a of t. So a of t is just a conversion factor here that converts the local speed of this light pulse from meters per second to notches per second, which is what dx/dt has to be measured in.

So this will be the speed. That means that the maximum distance that light will travel, still measured in notches in co-moving coordinates, will just be the integral of the speed. The integral of dx/dt is just Δx . So it would be the integral of $dx/dt dt$, integrating from 0 up to t_0 , the present time.

Now this is not the final answer that we're interested in. We want to know the present physical distance or the present proper distance of this object that's the furthest object that we can see. And the way to go from co-moving distances to physical businesses is to multiply by the scale factor. And we are interested in the distance today, so we multiply by the scale factor today, the present value of the scale factor. So the answers to our problem, which I will call l_p or sub [INAUDIBLE] of t. I want to get the word horizon into the subscript someplace. So I will call it l_p comma horizon, which means the physical distance to the horizon at time t is just equal to x_{max} that we have here times the present value of the scale factor.

So it's a of 2 [INAUDIBLE] times x_{max} . Or the final formula, just substituting in x_{max} , will be of a of t naught times the integral from 0 to t naught of dx/dt . I'm going to substitute c over a of t. Let me just remind you that [INAUDIBLE] variable integration should never have the same symbol as the limits of integration, because that just causes confusion. They're never really the same thing. So I called the limits t_0 . So it's perfectly OK to call the variable of integration t. In the notes, I call the value of the time that we want to calculate this as t, and then I use t prime for the variable of integration. Whatever you do, you should make sure that those are not the same. There's one variable that corresponds to the variable that varies from the initial time to the final time. And then there's also another value that represents the final time.

OK, so now all we have to do is plug in a of t is a constant times t to the $2/3$ into this formula and we have our answer. Notice that it does obey an important property. There's an a in the numerator and an a in the denominator. And that means that when we put in the formula for a of t , the constant of proportionality b will cancel out. And it must, as the constant's proportionality is measuring the notches or notches per seconds to the $2/3$ power but proportional to the notch. And the answer can't depend on notches because notches are not really a physical unit. But it works. That's an important check. So-- now it's just a matter of plugging in here, and maybe I'll do it explicitly. I'll leave out the b 's [INAUDIBLE] so you see the cancel. We have b times t zero to the $2/3$ times the integral from 0 to t zero, times c over b times t to the $2/3$ times dt .

The b 's cancel as I claimed. The integral of t to the minus $2/3$ is 3 times t to the $1/3$. We then subtract the t to the $1/3$ giving it the value t zero on the positive side, and then we subtract the value [INAUDIBLE] same expression at zero. But t to the $2/3$ when t is zero vanishes. So we just get t zero to the $2/3$ from the upper limit of integration. I'm sorry, to the $1/3$ power. We integrate minus $2/3$. We get t to the $1/3$. The t to the $1/3$ multiplies the t to the $2/3$ giving us a full one power of t .

So what we're left with is just 3 times c times t zero, which has the right units. It should have units of physical distance. Speed times time is the distance. And it has a surprising factor of 3 in it. The naive answer would have just been c times t zero, saying that the light at time t travels so it travels a distance c times t zero.

That would be true, as I said, in a stationary universe. But the universe is not stationary. It's expanding. And the fact it's expanding means that you expect this to be bigger than c times t zero. It means that at earlier times, things were closer. So the light can save time by leaving early and traveling a good part of the distance while distances are smaller than they are now. And it's a full factor of 3 .

So that is the horizon distance for a flat, [INAUDIBLE] universe. We can also, since we know how to relate t sub zero to the Hubble expansion rate, we can express the horizon distance if we want in terms of the Hubble expansion rate by just doing that

substitution. So this then becomes 2 times c times the current Hubble expansion rate inverse. So these are both valid expressions for the horizon distance in this particular model of the universe.

Any questions about the meaning of horizon distance? There is actually a subtlety about the meaning of horizon, which I should talk about. The initial value of the scale factor in our model is zero. It's t to the $2/3$, and t goes to 0, the scale factor goes to 0. Things are of course singular there, where you don't really trust exactly what the equations are telling us at t equals zero. But that's certainly how they behave. a of t goes to 0 as t goes to 0.

That means initially everything was on top of everything else. So if everything was on top of everything else, why is there any horizon distance? Couldn't anything have communicated with anything at t equals zero, when the distance between anything and anything was zero?

The answer to that is perhaps somewhat ambiguous. We of course don't really understand the singularity. We don't claim to understand the singularity. And therefore anything you want to believe about the singularity at t equals zero you are welcome to believe. And nobody intelligent is going to contradict you. They might not reinforce you, but they're not going to contradict you either, because nobody knows.

So it is conceivable that everything had a chance to communicate with everything else at t equals zero at the singularity. And it's conceivable that when we understand quantum gravity, it may even tell us that. We don't know.

What is still the case is that if you strike out t equals zero exactly, then everything is well defined. You can ask what happens if a photon is sent from one object to another leaving that object at time ϵ , when ϵ is slightly later than zero. And you can ask how long does that photon take to go from object a to object b? And that's really exactly the calculation we did, except instead of going down to t equals zero, you go down to t equals ϵ , where ϵ is the earliest time that you trust your classical calculations. Then you'd be asking how far is the furthest

object that we could see during this classical era, the era that starts from t equals epsilon? The only difference would be to put an epsilon there instead of 0.

And the answer-- you can go through it-- differs only by some small multiple of epsilon. And if epsilon is small, it doesn't change the answer at all. Physically what is going on is that if you go to very, very early times and look at two objects a and b and trace them back, the distance between them does get smaller and smaller as epsilon goes to 0. So you might think that communication would be trivial.

But at the same time as the distance is going to zero, you can calculate the velocities. h remember is going like 2 over $3t$. h is blowing up. The velocities between these two objects a and b are going to go to infinity at the same time that the distance between them goes to zero. So even though they will become very close, if one sends a light beam to the other, the other is actually moving away faster than the light would be moving. The light would eventually catch up, but the amount of time it would take for the light to catch up is exactly what this integral is telling us.

So there is no widespread communication that is possible once these equations are valid. They really do say that in the early universe, things just cannot talk to other things because the universe is expanding so fast. And the maximum distance that you can see is more than c times t zero, but less than infinity. Yes?

AUDIENCE: Why does it matter how far away the furthest galaxies we can see are now? Because we're seeing them as they were a long time ago when they were closer to us.

PROFESSOR: Yea, now we certainly are seeing them a long time ago when they were closer to us. That's right. I would just say that this is sort of a figure of merit. If you want to describe what you think the universe looks like now, you would assume that those galaxies that you're seeing billions of light years in the past are still there, and you'd extrapolate to the present. So it's relevant to the picture that you would draw in your mind of what the universe looks like at this instant, although that picture would be based on things you haven't actually seen.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yes?

AUDIENCE: Is the fact that this number is greater than ct zero proof that the universe is the actual space of the universe is expanding?

PROFESSOR: I don't think so because if you just have the objects moving and a space that you regard as absolutely fixed and then you ask the present distance of that object given that you can see a light pulse that was emitted by that object, it will still be bigger than ct , because the object continues to move after it emits the light pulse. So I don't think this is proof of anything like that.

I think I should add that certainly we think of this as the space expanding. That is certainly the easiest picture. But you can't have any absolute definition of what it means for the space to expand.

AUDIENCE: So you're saying it has moved three times the original distance it was when it sent out an impulse?

PROFESSOR: No it actually was zero distance when it sent out the light pulse, because the first object we in principle could see is an object for which the light pulse left it at t equals zero when it literally was zero distance from us. And the light pulse actually then gets further away from us and comes back and eventually reaches us. We have enough information here to trace it. And that's how it behaves.

OK any other questions? OK, next thing I want to do, and I don't know if we'll finish this today or not. But we'll get ourselves started. We'd like to now, having solved the equation for the flat universe for determining a of t , we would now like to do the same thing for the open and closed universes. And we'll do the closed universe first because it's a little bit simpler. I mean they're about equal, but we're going to do the closed universe first because it's first in the notes that I've written.

So we know what equation we're trying to solve. This is really now just an exercise in solving differential equations. So one has to be clever to solve this equation. So

we will go through it together and see how one can be clever and find the solution to it.

The equation is $\dot{a}^2 = \frac{8\pi}{3} \rho - k c^2$ over a^2 , with ρ of t being equal to ρ_i times a^3 over t^3 . And I'm writing i here. I could just as well write 1. It could be any time. ρ times a^3 isn't dependent of time. So you can put any time you want there. And the numerator is just a fixed number.

OK, so our goal is to solve this equation. The first thing I want to do is something which is usually a good thing to do when you are given some physical differential equation. Physical differential equations usually have constants in them like g and c^2 , which have different units which complicate things. And they don't complicate things in any intrinsic way. They just give you extra factors to carry around.

So it's usually a little cleaner to eliminate them to begin with by redefining variables that absorb them. One can do that by defining variables that have the simplest possible units that are available for the problem at hand.

And in our case, we have these complications that k is measuring in inverse notch squared, and a is meters per notch. We could simplify all that by defining some auxiliary quantities. So in particular, I'm going to define an \tilde{a} , which is sort of like the scale factor, but redefined by the square root of k . And this is the case when k is positive. I said we're going to do the closed universe first. I would not want to divide by the square root of k if k were negative. That would be confusing.

Now the nice thing about this is remember k has units of inverse notch squared. \tilde{a} has units of meters per notch. That means the notches just go away. So this has units of length only. And that was the motivation for dividing by the square root of k , to get rid of the notch.

Similarly, time has units of meters per second. I'm trying to minimize the number of

distinct physical quantities that we have in our problem. So I'm going to define a \tilde{t} which is just equal to c times t . This is of course no difference in just saying the working units is c is equal to 1, which a lot of times is something people say. This is the same thing except I'm a little bit more explicit about.

So \tilde{t} now is units of length also. So the idea is to translate everything so that everything is the same units, which would be meters or whatever physical unit of length you want to use. OK, now I'm just going to rewrite the Freedman equation using these substitutions. And to make way for the substitutions, I'm first just going to divide the Freedman equation by $k c^2$. This is k to the $3/2$.

OK now when I divided by $k c^2$, the $k c^2$ over a squared there became-- I'm sorry. I'm doing more than dividing by $k c^2$. I'm dividing by $k c^2$ and multiplying by a^2 . So I have turned that last term into 1 by multiplying by its inverse. On the left hand side, the 1 over a squared went away when I multiplied by a^2 . So I just have a dot squared divided by $k c^2$ so far. We'll simplify that shortly. And the middle term, I took the liberty of multiplying by the square root of k , and then we have a k to the $3/2$ here. If we absorb this into that, it would just be the $k c^2$ factor that we divided by. And the a^2 has been also divided into two pieces, a cubed and a. So together these two factors make up the factor of a^2 that we multiplied that term by. So it's the same thing we had, just multiplied by the common factor.

And now the nice thing is that this is, in fact, our definition of $\tilde{d}a$ over dt . The c turns the dt into a $\tilde{d}t$, and the 1 over k turns the da into $\tilde{d}a$. So the left hand side now is simply $\tilde{d}a$ over $\tilde{d}t$ squared. And the right hand side, ρ times a^3 remember is a constant. So the only thing that depends on time on the right hand side is the a divided by μk , which I've isolated here. Earth That's a $\tilde{d}a$.

So I'm going to take all of this, which is a constant, and just give it a name, constant. So this term becomes a constant, we'll call 2α divided by $\tilde{d}a$. And then we still have minus 1. And this constant, which I'm calling α , is just all of this factor

except for a factor of 2. So it's 4π over $3g\rho a^3$ divided by c^2 .
a tilde because we had a cubed divided by k to the $3/2$, and that's a tilde cubed.

So I've just rearranged things. But now everything has the units of length. a tilde has units of length. t tilde has units of length. This is dimensionless, which is good because that's also dimensionless. Alpha, if you work out all this stuff, has units of length. a tilde has units of length. This is length divided by length, dimensionless.

We haven't really changed anything significant. But at least as far as keeping track of factors, that equation is the one we'll solve. And the factors are now absorbed into the constant alpha, which is the only thing we have to worry about. And we don't need to worry about that until the end. Yes?

AUDIENCE: [INAUDIBLE]

PROFESSOR: As long as these two are evaluated at the same time, it doesn't matter.

AUDIENCE: [INAUDIBLE]

PROFESSOR: That's right. One does need to remember that these two are to be evaluated at the same time, whatever it is. And the product does not depend on time. So I didn't write the arguments. I could have, and I could have just put in an arbitrary time. But it would be the same time for the ρ and the a tilde.

I will do that. It will be ρ of some t one times a^3 tilde, evaluated the same t one for any t one. This product is independent of time.

OK, now that is the kind of equation where we could move things from one side of the equation to the other to reduce it to doing ordinary intervals. So I can multiply by da tilde and divide by the expression on the right hand side and get an expression that says that dt tilde is equal to da tilde divided by the square root of 2α over a tilde minus 1.

And since this has an a tilde in the denominator of the denominator, I'm going to multiply through by a tilde to rationalize things. So I'm going to rewrite this as da tilde over the square root of $2\alpha a^2$ tilde minus a^2 tilde squared. OK this looks

better than we've had so far.

Now in principle, if we imagine we can do that integral, we can just integrate both sides. We get an equation that says \tilde{t} is equal to some expression involving the final value of a . So we're going to imagine doing that. And we will actually be able to carry it out.

When we integrate, there are two ways you can proceed. When I solve the analogous problem for the flat case with the t to the $2/3$ it was a much simpler equation. But you might remember that at one point, I had an equation where I calculated the indefinite integral of both sides, and then I got a constant of integration, which I then argued should be set equal to zero if we wanted to find the zero of time to be the zero of a .

The situation is really exactly the same, but to show you an alternative way of thinking about it, this time I'm going to apply definite integrals to both sides. And if you apply a definite integral to both sides, the thing to keep in mind is that you should be integrating over the same physical interval on both sides. So on the left hand side, I'm going to integrate from zero up to some \tilde{t}_{final} . \tilde{t}_{final} is just some arbitrary time, but I'm going to give it the name \tilde{t}_{final} .

And that integral of course we can do. It's just \tilde{t}_{final} . And that should be equal to the integral of the right hand side over the same period of time. But the right hand side is not expressed as an integral over time. It's expressed as an integral over \tilde{t} . So we have to ask ourselves, what do we call the integral over \tilde{t} that corresponds to the integral over time from 0 to $\tilde{t}_{\text{sub } f}$.

And the lower limits should match. And we know what we want \tilde{t} to be at \tilde{t} equals zero. We're going to use the same dimensions we had in the other case. We're going to define the zero of time to be the time when the scale factor is zero. So \tilde{t} equals zero should correspond to \tilde{t} equals zero.

So to get the lower limits of integration to correspond to the same time, we just put zero here. And zero here doesn't mean time zero. It means \tilde{t} equals zero. But

that's what we want. And for the upper limit of integration, that should just be the value of a at the time t sub f . So I will call that a sub f , where I might make a note on the side here that a sub f is e tilde of t tilde sub f . It's just the final value of a tilde, where final means anytime I choose to stop this integration. The interval is valid over any time period.

So these are moments of integration where the only thing that's new on this line-- now I just copy from the line above, 2α times a tilde minus a tilde squared. So t tilde f is equal to that integral. And our goal now would be to do that integral. And if possible-- it won't quite be possible. We can see how close we can come. But if possible we would like to invert that relationship that we get from doing this integral, to determine a tilde as a function of t tilde.

That's what we would love to do. It turns out that's not quite possible. But we will nonetheless be able to obtain something called a parametric solution to this problem. And we'll stop now. But I will tell you next Tuesday after the quiz how we proceed here to get a solution to this problem.