PROFESSOR: I think it's a good idea to review where we were because we're kind of in the middle of a discussion. We're actually on part three. And probably, there will be four parts all together of our discussion of homogeneous expansion. So I have a few slides to just review where we were last time.

We were building a mathematical model of our homogeneously expanding universe. And we modeled it as a finite-sized sphere where we promised that in the end we will take the limit as the size of that sphere goes to infinity and fills all the space. But it started with some initial maximum, R max i, i for initial.

We arranged for it to have a uniform density, rho. We started at a time called t sub i. There were some initial maximum radius, R max i. And we also set this up to exhibit Hubble expansion. And we're going to try to calculate how it evolves from there. But we're going to start it Hubble expanding.

And Hubble expanding means that we're starting with every particle having an initial velocity which is a constant, H sub i, the initial value of the Hubble expansion rate, times the vector r. That is, the vector that goes from the origin to the point where the particle is located. And that corresponds to Hubble expansion centered on the center of our sphere. So these are our initial conditions. We just put them in by hand because we think they form a good model for what our universe looks like.

And then, the laws of evolution should take over to govern what's going to happen later. And since we haven't studied general relativity, we'll be using Newton's law of gravity to discover how it behaves. But I promised you at the beginning that we will, in fact, get exactly the same equation that we would have gotten had we used general relativity. And we'll talk later today, probably, about why that's the case.

So this is the initial setup. Any questions about the initial setup?

OK, then we derived a lot of equations. And everything really follows from the statement at the top here, which is understanding what Newton tells us about the gravitational field of a spherical shell. We could always think of our solid sphere as being made up of shells. So if we know how a shell behaves, we know all we need to know.

And Newton told us. He told us that inside a spherical shell, the effect of gravity, which I'm describing in terms of the gravitational acceleration vector, little g, inside a shell, the gravitational field is exactly 0. The force is coming from all different parts of the shell, pulling outward, cancel each other. And the net force on any object anywhere inside the shell is exactly 0.

Outside, the entire shell acts exactly as if it were a single point mass located at the origin with the same total mass, m. So incredibly simple. It's hard to believe it's so simple, but it is.

By the way, if you use Gauss's law of gravity, it becomes very obvious that those statements are the right statements. Newton know about Gauss's law of gravity, so Newton derived those statements by brute force integration, which is more of a tour de force. But something Newton was capable of doing it, and he did.

To describe how the system is going to evolve, moving onward, we introduce something that's a little bit complicated, a function r, which is a function of two variables, r sub i and t. And r represents the radius at time t of the shell that was initially at radius r sub i.

And our goal here is not just to keep track of the particle on the outside, which is for example, what Ryden does in her textbook. Ryden assumes that everything stays homogeneous. And then if you follow what happens to the outer edge, you know everything. But we're not going to make that assumption.

We're going to conclude that it remains homogeneous, but we're going to derive it. Which means that we need to know the motion of every particle inside the sphere to be able to tell if it's going to stay homogeneous. And that's why we're introducing this more general description where r is a function of r sub i. So that function of the extra variable will tell us how every particle moves as the system evolves.

We know that we're going to maintain spherical symmetry because we start with spherical symmetry and the force law respects spherical symmetry. So we're building that in from the beginning. We're not allowing things to depend on angular variables theta or phi. But assuming spherical symmetry, motion just is described entirely by giving the r-coordinate of each particle. And this function r of r i t does that-- exactly that.

Then, we said that at a given radius, this description about shells tells us that the shells that are outside that radius don't do anything at a given radius. But the shells that are inside act like a point mass, as if it was all at the origin. So to understand how a given shell is going to evolve, all we really need to know is the total mass inside that shell. And that's given by M of r sub i, the mass inside the shell at radius r sub i. And that's just the volume of the shell initially times the initial mass density.

As these shells move, the total mass inside a shell will remain exactly the same as it was as long as there's no crossings of shells. Now, the shell crossing issue is hard to talk about. But in the end, it just doesn't happen so you don't need to worry about it.

But the argument was that initially we know the shells are moving apart from each other because we built in this Hubble expansion where everything is moving away from everything else. So if shells are ever going to cross, they're not going to cross immediately. It will take some time for these velocities to reverse, and the shells that were moving apart to move together and hit. So there's unambiguously at least a period of time where there are no shell crossings. And we could write down the equations that describe the motion during this period where there are no shell crossings.

Now, if there was going to be a shell crossing, the equations that we're writing down would have to hold right up until the time of that first crossing. Because as long as there's no crossings, our equations are valid. Which means that if there was going to be a shell crossing, the equations we're writing down had better show it. Because the equations that we're writing down have to be valid right up until the instant of any possible shell crossing.

And what we're going to find is that the equations are going to lead to just homogeneous evolution where there are no shell crossings. And therefore, that's the conclusion. If there were going to be any shell crossings, these equations would have to show it. They don't show it, so there are no shell crossings. So it's a complicated paragraph, but the bottom line is simple- we can ignore shell crossings.

And that means that the total mass inside of any shell will remain exactly constant with time given by its initial value. And this formula is the initial value. And therefore, it holds at all time. Any questions about that? Any questions about the shell crossing issue?

OK, good. So whoops, sorry about that.

So M of r sub i is the mass inside the shell at radius r sub i. And then we can write down-- now we use Newton's law of gravity directly. We can write down the acceleration of a given particle in terms of its radius r and its initial radius r sub i. Its initial radius determines how much mass is inside. M of r sub i is independent of time.

But the actual radius it's at determines how far away it is, or describes how far away it is from the origin. And that's the 1 over r squared that appears in the force law. So we have the time dependent r in the denominator and the time independent initial r sub i that appears in the numerator. And it's all proportional to a unit vector r hat pulling everything radially inward because of the minus sign in front. So gravity is pulling everything inward, which is what we'd expect.

So this formula is the key formula. It's a vector formula, but we know that all the motion is radial. So all we really have to keep track of is the radius as a function of time. So we can turn this vector formula into a formula for little r itself. Just the radius number, the radial coordinate. And we get r double dot is minus 4 pi over 3 G r sub i cubed rho sub i, taking the formula for M of ri from the line above divided by r squared.

And the r in this formula-- I didn't write the arguments, but it means this function r, which is a function of two arguments, r sub i and t. So this differential equation now governs our entire system and tells us everything we need to know or everything we can know about the actual dynamics.

But to solve a differential equation of that sort, a second order differential equation, we need initial conditions. And we already described the initial conditions in words. Now we have to just figure out what those initial conditions are saying about r and r dot.

And the answer is straightforward. We argued it last time. The initial conditions are that r at time t sub i is just r sub i. That was really the definition of r sub i in the first place. And r dot is just H sub i times v coming from the formula we had for the initial velocities.

So these three equations, the two initial conditions and the differential equation, lead in principle to a mathematical solution that's completely unique and determined by those equations. And our goal now is just to figure out what that solution looks like.

And we discovered a marvelous scaling property. That is, instead of talking about r, we divided r by r sub i and defined a new function, which we initially called u. Initially, thinking of it as a function of these two variables. We can write down new equations for u. And those equations end up not having any r sub i's in them at all. And once we realized that, we realized we don't need to call it u of r sub i and t. It's really just a function of t.

And at that point, we renamed it because we also realized that it actually is our old friend the scale factor, a of t. So a of t is just r of r sub i and t divided by r sub i.

And then the reason this is a scale factor is seen most clearly from that equation, which is really this equation just rearranged. The physical distance of a particle from the origin is equal to the scale factor times r sub i where r sub i plays the role of the coordinate distance. r sub i is a time-independent measure indicating which shell you're talking about.

So the equations for a then are the equations for u, which are the equations for r just divided by r sub i. And we can write down what those equations are.

We have a differential equation for a and two initial conditions where r sub i has dropped out all together. The differential equation is given immediately from the one we had up here, but we could also we write it in terms of what rho of t is.

I didn't write the equation here because I guess I was running out of room. But we also figured out how rho of t behaves. And it behaves in the obvious way. As the space expands, the density just goes down as the volume. And the volume grows like a cubed, the cube of the scale factor, because volume's proportional to radius cubed. So rho of t is just rho sub i divided by a of t cubed.

And putting that in, we can go from that equation to this equation. And this equation makes no reference anymore to the initial time. It's just an equation for what deceleration you see, what value of a double dot you see, as a function of the mass density and a itself. OK, any questions about any of those differential equation manipulations? Yes.

AUDIENCE: In the homework, it says that this equation is not entirely general. We can't use it in all cases. Whereas, the other version that we get from the energy conservation is completely OK?

PROFESSOR: That is correct, yes.

AUDIENCE: It says it in there, but why is that?

PROFESSOR: Why is it true? Well, as long as you have only non-relativistic matter, which is what we're talking about here, both of these-- this equation is golden and so is the equation we're about to talk about the derivation of, the conservation of energy equation. So as long as we have the context in which we derived it, it's completely valid.

But on the homework set, we're talking about more general cases. We gave a different formula for the scale factor, which corresponds to a different situation in terms of the underlying materials that are building that universe. And where the change occurs is when one introduces a nonzero pressure. This gas of particles that we're talking about is just non-relativistic particles moving with the Hubble expansion. There's no internal velocity which generates a pressure. And it's pressure that makes a difference.

This formula assume zero pressure. We will learn later how to correct it when there's a nonzero pressure. And the other formula doesn't depend on pressure, so it's valid whether there's pressure or not.

But at the moment, we have no real way of knowing that. We'll talk later about why that's true. Other questions?

No. OK, great. OK, one more slide here, not too much on it. At the end of the last class, we took the equation that I just wrote on the previous slide. I just copied it to this slide, the equation for a double dot, written in terms of the initial mass density. And discovered that it can be integrated once to produce a kind of a conservation of energy equation.

And all you do is you start with this, write it by putting everything on one side of the equation, a double dot plus 4 pi over 3 G rho i over a squared equals 0. And then, multiply the whole equation by a dot. And a dot is called an integrating factor. It turns the expression into a total derivative.

So once you write it this way, it is equivalent to dE dt equals 0, where e is just defined to be this quantity that would have better as a triple equal sign. e is just defined to be that quantity. And if you then write down what dE dt means, it means exactly that. So it's the same equation.

So given our second order equation, we can write down a first order equation, which is that E is equal to a constant. And we commented last time that the physical interpretation of E is-- I'm not sure what to say. There are multiple physical interpretations of E is probably what I want to say.

And one physical interpretation is if you multiply by the right factors, it does describe the actual energy of a test particle just on the boundary of our sphere, on the outer boundary. It doesn't really describe directly the total energy of a particle inside that sphere because calculating the potential energy of a particle inside the sphere is more complicated. And it doesn't give you the simple answer. So it doesn't really describe particles on the inside, except you could argue that if you-- talking about a particle on the inside, the particles outside of it don't matter. And you pretend they're not there. And then it does describe the energy. That is, you could think of any particle as being on the outside of the sphere and ignore what's outside it. But that's extra sentences that you have to put in.

On the homework, you will also discover that for this finite-sized Newtonian sphere, there's certainly a well-defined Newtonian expression for the total energy of the sphere. And that's also proportional to this E. So by multiplying it by different constant, you can turn it into the total energy of the sphere. So it's actually related to energy and it's definitely conserved. Those are the important statements to takeaway.

And that's where we left off last time. And we'll pick up from there now pretty much on the blackboard for the rest of lecture. Are there any further questions about these slides?

OK. In that case, we will go on.

The first thing I want to do is to take the same conservation law that we have up there and rewrite it in a way that's more conventional. And perhaps, more useful. But certainly, more conventional.

We started with knowing that a quantity called E is conserved. And it's equal to 1/2 a dot squared minus 4 pi over 3 G rho i over a. OK, then we also know that rho of t is equal to rho sub i divided by a cubed of t, which just says that matter thins out with the volume which grows as the cube of the scale factor. And that can be used to manipulate this equation.

For reasons that will become clearer in a minute, I'm just going to manipulate this equation by multiplying it by 2 over a squared. Just because this will get me the equation I'm trying to get to.

So if I multiply the left-hand side by 2 over a squared, I have to multiply the right-hand side by 2 over a squared. The 2 cancels the half. The first term becomes a dot over a squared. And you might remember the a dot over a is the Hubble expansion rate, so it has some physical significance.

And then, minus the 2 turns the 4 pi over 3 into 8 pi over 3. And the a squared multiplies the a to make an a cubed. And then here, we have rho i over a cubed, which in fact is just the current value of rho. So we can rewrite this as a dot over a squared minus 8 pi over 3 G rho. No a's anymore on the right-hand side.

Well, on a's anymore in this term. OK, now the convention that brings our notation into contact with the rest of the world. Nobody talks about E, by the way. That's just my convention. But to make contact with the rest of the world, the rest of the world talks about a number called little k, lowercase k. And it connects to our notation by being equal to minus 2E divided by c squared.

And with that connection, we can write our conservation law in what is the standard way of writing it, at least in many textbooks.

And that's it. So I put a box around it. And this equation was first derived by Alexander Friedmann using general relativity in 1922. And it is, therefore, usually called the Friedmann equation.

Alexander Friedmann, by the way, was a Russian meteorologist by profession. But as a meteorologist, he knew a lot about differential equations. And when general relativity came out, he got interested in it and was the first person to derive using general relativity the equations that described an expanding universe. And he wrote two famous papers-- now famous papers-- in 1922 and 1923. One of them talking about the system of equations where k is positive and another talking about where it's negative. I forget which order they were in. But they correspond to open and closed universes, which we'll talk about more in a few minutes.

Now, I just should remind you to have our equations together. We also had the all-important equation for a double dot. And as I was just describing an answer to a question, we don't know yet how to generalize this to other kinds of matter besides the non non-relativistic dust that we just derived them for.

They're certainly both correct for our non-relativistic dust. But when we try to generalize them, what we'll find is that the top equation will remain true exactly for any kind of matter, while the bottom equation assumes that pressure equals 0.

Now, the standard terminology is to call the top equation the Friedmann equation. In fact, both of these equations, with this one including the pressure term that we don't have yet, appeared in Friedmann's original papers. So I usually refer to these two equations as the Friedmann equations-- plural. But many textbooks refer to just the top one as the Friedmann equation and don't give a name to that equation, which is also OK if you want. Yes.

AUDIENCE: Didn't we get the top equation from the bottom equation?

PROFESSOR: We did. That's right. So how does that jive? The answer is-- and we'll be coming to this later. But the answer is that when we got the top equation from the bottom equation, we used that equation.

And this equation will no longer hold when there's a significant pressure. And in fact, when we derive it later-- I forget what order we'll do things in. But we'll make sure that all three of these equations are consistent when we include pressure.

The reason the top equation changes if you include pressure-- may not be obvious. But if I tell you why it happens, it will become obvious. The top equation looks like it's just how things thin out. Like a cubed. But rho is the total mass density. And relativistically, it's equivalent to the energy density. If you just multiply by c squared, that becomes the energy density by the E equals mc squared equality. So it's a question of how much energy there is inside this sphere or box.

And if you imagine a box changing size, if it's filled with a gas with a positive pressure, as that box changes size, the pressure does work on the boundary. If you think of it as a piston. And if we have a positive pressure and a gas expands, it loses energy. And relativistically, that means it has to also lose mass.

The total mass inside the box does not remain constant as it expands, which is the idea that we use when we derive that rho i over a cubed. So rho i over a cubed is the right behavior for the total mass density or energy density for a zero pressure gas. But when you include pressure and take into account relativity, that's not the right formula anymore.

AUDIENCE: The top one, it just cancels out somehow?

PROFESSOR: Well, the pressure ends up canceling out, so that this ends up still being true and this ends up being different. And we'll see later how it happens. I just want to indicate where the changes are going to be. We'll see what the changes are when we get there.

AUDIENCE: What happened to the third factor of a in the second equation?

PROFESSOR: There's a factor of a missing, sorry. Yes, thank you very much. It's important to get the equations right. That wasn't Friedmann's equation. That was Alan's equation.

Now it's Friedmann's equation. He got it right. OK, any other errors or questions to bring up?

OK. Now is probably a good time to talk about the question of why we were so fortunate to discover that the Friedmann equation that we derived agrees exactly with general relativity. There is a simple reason for it. I don't think it's an accident at all.

The reason for it, as I understand it, is that we assume from the beginning-- and we would be assuming this whether we're talking about the Newtonian calculation or the corresponding general relativity calculation. We assume from the beginning that we are modeling a completely homogeneous system, where every part of it is identical to every other part.

And once you assume that homogeneity, it means that if you know what happens in a little box, a meter by a meter by a meter say. If you know what happens in a little box, you know what happens everywhere because you assume that what happens everywhere is exactly the same as what's happening in that box. So that implies that if Newton is right for what happens in the box, Newton has to be right for what's happening in the universe.

And we do expect Newtonian physics to work on small scales, scales of a meter, and small velocities. The Hubble expansion of a meter is negligible. So we expect Newton to give us a proper description of how the system is behaving on small scales. And the assumption of homogeneity guarantees that if you understand the small scales, you also understand the large scales.

So I think we are guaranteed that this had better give us the same results as general relativity or else Newtonian physics is not the proper limit of general relativity. But it is. We would not accept general relativity if it did not give Newtonian physics for small scales and low velocities. So we expect to get the same answer as Gr and we do. This is exactly what Gr would give. And this is exactly what Gr would give also for the case where there's no pressure-- the case we're doing. OK, any questions about that?

OK, next item. I would like to say a couple words about the units in which we're going to define these quantities. So far, in our mathematical model, r and r sub i are distances. And therefore, they're measured in whatever units you're using to measure distances. And I'll pretend we're using SI units. So I'll say meters.

We could use light years, or whatever. It doesn't really matter. But they're both measured in ordinary distance units.

And earlier when we talked about scale factors and things, I told you that many books do it this way. Think of both the co-moving coordinates and the physical distances as being measured in meters and the scale factor being dimensionless. But I told you I don't like to do it that way because I think it's clearer to recognize that these co-moving coordinates don't have any real relationship to actual distances. At least not as time changes.

So for me, it's better to have a different unit to describe the co-moving coordinate systems. So I would like to introduce that here. And all I need to do really is say that let the unit of r sub i be called a notch.

Now, we already called it a meter, but that doesn't really mean that a meter is the same as a notch. Because when we called it a meter, we were not really taking into account the fact that it's only a meter at time t sub i.

So another way of describing this definition, which might not sound like we're trying to redefine the meter, is to say that the statement r sub i equals 5 notches, which is the kind of statement I'm going to make now because I'm only going to talk about r sub i measured in notches.

When I say r sub i is 5 notches, that's equivalent to saying that the particle labeled by r sub i equals 5-- 5 notches-- was at 5 meters from the origin at time t sub i. So giving the value of a co-moving coordinate in notches tells you exactly what distance it was from the origin at time t sub i.

Now, the reason why I don't use this to just say, well, why don't we call it meters, is that we're now going to forget about t sub i. If you look at equations we have, t sub i no longer appears in any of them. t sub i was just our way of getting started. And we could have started at any time we wanted.

And once we have these equations, we can talk about times earlier than t sub i, times later than t sub i. And there's nothing special about t sub i anymore. But r sub i we're going to keep as our permanent label for every shell, which means for every particle there will be a value of r sub i attached to that particle which will be maintained as the system evolves. And it will be clearly playing the role of what we've called the co-moving coordinate.

And therefore, we will want to keep r sub i and we'll want to call its unit something. And I'm just saying I'm going to call them notches. You can call them meters if you want, but I'm going to call them notches.

So using this language, I'm not changing any of the equations that we wrote. So we will still have r being equal to a of t times r sub i. But now, r will be measured in meters. a of t I will think of as meters per notch. And r sub i will me measured in notches.

And the scale factor a of t will be playing the same role it played ever since we introduced the word scale factor. It just means that when the scale factor doubles, all distances in our model double exactly.

Now that we have the sort of new system of units, I just want to work out the units of an object where the units are not all obvious. What are the units of this thing that we've defined that we called k, which is related to this thing that we defined that was called E, where E was not really an energy? But we can work out what k is.

I'm using square brackets to mean units of. k can be thought of as being defined by this equation. Or in any case, this equation certainly must be dimensionally consistent because we derived it and you didn't point out that I made any mistakes, so I must not have. So the units of kc squared over a squared should be the same as the units of a dot squared over a squared.

And if I multiply through to write units of k-- to relate them to a dot, we get the units of k have to be the same as the units of 1/c squared times a dot squared. The a squareds in the denominator here just cancel each other. So the units of k has to be the same as the units of 1/c squared a dot squared. And we know what they are.

The units of 1/c squared is second squared per meter squared. Meter per second squared, but upside down. That's the 1/c squared factor. And a dot is meters per notch per second because of the dot. So meters per notch would be a. a dot would have an extra second. And the whole thing gets squared because it's a dot squared here.

And you see then that the meter squares cancel. The second squareds cancel. And we just get the peculiar answer 1 over a notch squared.

Now, there's probably not a lot of intuition behind that. But what it clearly does say is that we can make k change its numerical value by changing our value for the notch. And that's important to know. And it's a clear consequence of what we just did.

So now we can talk about different conventions that people use for defining the notch. And hence, k, which are clearly related to each other we've now learned.

So first of all, in the construction that we just did-- starting out with our sphere, and letting it grow, and defining the maximum sphere as r max comma i and so on. In that construction, our initial value of t-- excuse me, our initial value of a, a of ti, was one. And the way we first did it where all lengths were measured in meters. And with our new definition that r sub i is measured in notches so that a is meters per notch, it means that in the system we just had, a of t sub i was 1 meter per notch.

And since we already did this, we don't really want to change it. But I point out that t sub i, if you look at our equations, survives in only one place, which is in this equation. It has disappeared from every other equation we're going to be keeping. So we are perfectly safe in just forgetting about this equation. Or if we want to remember about it, we could just say, well, yeah, t sub i had some significance. It was the time at which the scale factor happened to have been equal to 1 meters per notch. But otherwise, it has no importance. There's a different time when the scale factor was 10 meters per notch, or 1 light year per notch.

So since t sub i is of no relevance whatever, we can safely forget the above equation. Or we could think of it as the definition of t sub i, where we don't care anything else about t sub i. So it was just some symbol that was used in some earlier calculation.

So we now just have a scale factor a of t. And we can talk about how we might normalize it. And there are basically two important conventions that are in use in textbooks.

Some textbooks, which include Barbara Ryden's textbook that we're using, define a of t to be equal to 1, which I will call 1 meter per not much today. So a of t is just defined to be 1 today as a common notation. It's notation that Barbara Ryden uses. And that makes a notch equal to a meter today.

There's another common convention, which is to recognize that since k has units 1 over notches, we can make k any value we want without changing any of the physics, just changing our definition of the notch, which is up for grabs. Nobody has yet defined the notch. We're defining it now.

So we can choose the definition of a notch to make the value of k something simple. And the obvious choice for the simplest real number that you could imagine that's not 0 is 1. So we could choose the definition of the notch to make k equal to 1. Except that we can't change the sign of k. The sign of k makes important differences in this equation. And we don't want to make the definition of a notch negative or imaginary, I guess, is what you need to change the sign of k. So as long as notches are real, we can only change positive k's to different positive k's and negative k's to different values of negative k.

So the convention would be that when k is not equal to 0-- it can be 0 as a special case. But when k is not equal to 0, define the notch so that k is equal to plus or minus 1. I would say that this convention is more common than that convention of the books that I've read in my lifetime, but both conventions are in use.

And one sees from this dimensional relationship that one can certainly choose a notch to make k if it's nonzero have any value you want of the same sign. And that means you can always make k plus or minus 1 if it's not 0.

The books that use this as their convention, generally speaking, leave the notch undefined when k equals 0. Undefined means arbitrary.

And there's no problem with that because k and the notch never really appear in final physical quantities. The notch always was just your choice of how to write down your co-moving map of what the universe looks like. OK, any questions about these funny issues involving units?

OK, good. The next thing I want to do now is to start talking about solutions to this equation. And I guess I'll leave the equation there and start on the new blackboard.

OK, I'm going to rewrite the equation almost the way it was written on the top there. In fact, exactly as it was written on top there. I'm going to rewrite the equation as E equals 1/2 a dot squared minus 4 pi over 3 G rho sub i over a. And the reason I'm writing it in this way rather than any of the other six or seven ways that we've written it, is that this way the only time-varying thing is a itself. And if we want to talk about what the differential equation tells us about the time variation of a, it helps a lot if we're writing an equation where a of t is the only thing that varies with time. And this is at least one way of doing that. So in particular, I used rho sub i rather than rho.

So the behavior of this equation might very well depend on the sign of E. And we'll see that it does. And if we think it might depend on the sign of E, we realize from the beginning that E could be positive, negative, or 0. There are those three cases. So those are the cases we want to look at. E can be positive, negative, or 0. So we'll take them one at a time.

It will help to make things completely obvious to rewrite this as an equation for a dot squared. I'll multiply by 2. And write this as equation a dot squared is equal to 2E plus 8 pi over 3 G rho sub i over a. 8 pi instead of 4 pi because we multiplied by 2.

And we notice that this term, proportional to rho sub i over a, is unambiguously positive. We're not going to have any negative mass densities in our problem. There's no way that can happen. And a is always positive. So the right term is positive. a dot squared had better be positive because it's the square of something. E could, in principle, have either sign. And we'll talk about both cases, or the 0 case.

But if we start with the case where E is positive, just to consider these three cases one at a time. So suppose E is greater than 0. And remind you that E and k had opposite signs. So that would imply that k was negative. So if we start by considering the k negative case or the E positive case, then we see that we have a positive number here and a positive number there. So they will add up to always give us a positive number. a dot squared will always be positive. And it will just mean that a dot will be positive. Square root of a positive number is a positive number. At least it's only the positive square root that matters here. So a will just keep growing forever in this case.

a dot squared will never fall below 2E. So it will be a lower bound to a dot squared. And that means there will always be a minimum expansion rate that the universe will have. So in this case, a increases forever. And that's called an open universe. And it's one of the three possibilities that we're going to be investigating here.

Next case is E less than 0, which is the more common notation of k means k is greater than 0. In this case, if you think of E as an energy, which it really is, it means we have less than 0 energy. Which means that we basically have a bound system. And the equation tells us that it acts like a bound system. We don't have to rely on that intuition, but that is the right intuition.

The equation up there tells us that if E is negative, the total right-hand side had better be positive because the left-hand side is positive and the left-hand side cannot go negative. But this term is going to get smaller and smaller as a increases. And as this term gets smaller and smaller, it runs the risk of no longer outweighing this term and giving possibly a negative answer.

And what has to happen is a cannot get any bigger than the value it would have where the right-hand side would vanish. So a continues to grow because a dot is positive. This gets smaller and smaller until this term equals that term in magnitude. And then, a dot goes to 0.

What happens next is not completely obvious from this equation, but it means that we have an expanding universe that's reached a maximum size and then stopped. Then, what is actually obvious is from this equation is that it will start to collapse. So this case corresponds to a universe that reaches a maximum size and then turns around and collapses.

So a has a maximum value. And we can read off from that equation what it is, a max is just the value that makes the right-hand side of that equation 0, which is minus 4 pi G rho sub i divided by 3E. And remember, E is negative for this case, so this is a positive number. So a has some positive maximum value. Reaches that value, and then turns around and collapses. Yes?

AUDIENCE: Sorry, I don't know if you said this already, but since a dot squared is equal to some quantity, when you solve for a dot, you can have positive and negative solution. Why do we discount the negative solution?

PROFESSOR: OK, very good question. The question if you didn't hear it is, why do we discount the negative solution when we have an equation for a dot squared? Couldn't a dot be positive or negative?

And the answer is it certainly could be either. And both solutions exist as valid solutions to these equations. But we started out with an initial condition that a dot was positive. Our initial value of a dot was H i. And once it's positive, it can't change sign according to that equation. Except by going through 0, which is what we're talking about now. But it will only change sign when it goes through 0.

So it reaches a maximum value, then it does collapse. And in the collapsing phase, that same equation, a dot squared equals the right-hand side, holds where it would be the negative solution that describes the collapsing phase.

So the verbal description of what's happening here is that the universe reaches a maximum size and then collapses. And it collapses all the way to a equals 0 in this model. And that's often called the Big Crunch for lack of a better word. The Big Crunch being the collapsing form that corresponds to the Big Bang, which is the instant which all this starts.

And this was called an open universe. As you could probably guess, this is called a closed universe.

OK. And now finally, we want to consider a case where E is not positive and not negative. The case that we're left with is E equals 0. And that's called the critical case. So the critical value for E is E is equal to 0. And that means that k is equal to 0 as well. It implies k equals 0.

And notice that this is a special case. E is a real number. It can be positive, negative, and 0 is just a particular value on the borderline between those two. For the people who are in the habit of rescaling notches so that k is always plus 1, minus 1, or 0, it makes it sound like there are three totally distinct cases. But that's only because of the rescaling that those people are doing.

If you keep track of E as your variable, which you certainly can, you do see that the flat case, E equals 0-- the critical case-- is really just the borderline of the other two. And it's therefore, arbitrarily close to both of the other two. It really is where they meet.

But working out the equations, we have in this case a dot over a squared is equal to 8 pi over 3 G rho. And in general, it's minus kc squared over a squared. But we're now considering the case where that vanishes. And that means we have a unique value for rho in terms of a dot over a. And at this point, it's worth reminding ourselves that a dot over a is just H. So this is H squared.

So for this critical case, the case that's just on the borderline between being open and closed-- and we'll be calling it flat. For this critical case, we have a definite relationship between rho and H. So rho has to equal what we call the critical density, which you get by just solving that equation. And it's 3H squared over 8 pi G.

And we see, therefore, that rho being equal to rho c is this dividing line between open and closed. And if you think back about the signs of what we had, what you'll see is that rho bigger rho c is what corresponds to a closed universe. Rho less than rho c is what corresponds to an open universe. And rho equals rho c can be called either a critical universe or we'll be calling it a flat universe.

And the meaning of the word "flat" will be motivated later. For now, these words--

open, closed, and flat-- refer to the time evolution of the universe. We'll see later that it's also connected to the geometry of the universe, but we're not there yet. Then the word "flat" will make some sense. Yes.

AUDIENCE: How do we know there's not some very large entity, like some cluster of black holes or something that renders all of this not applicable to our universe?

PROFESSOR: OK. The question is, how do we know there's not some humongous perturbation, some huge collection of black holes that renders this all inapplicable to our universe?

The answer is that it works for our universe. That is, observationally we can test these things in a number of ways. Tests include calculations of the production of the light chemical elements in the Big Bang. Tests include making predictions for what the cosmic background radiation should look like in detail. And those tests work extraordinarily well. So that's why we believe the picture.

But you're right, we don't have really direct confirmation of most of this. And if there was some giant conglomeration of mass out there someplace, it might not have been found yet. But so far, this picture works very well. That's all I can say. And there really is quite a bit of evidence. We'll maybe talk more about it later. Any other questions?

OK. So having understood the importance of this critical density, it might be nice to know what the value for the critical density for our universe is. And we can calculate it because it just depends on the Hubble expansion rate, an the Hubble expansion rate has been measured.

So if we try to put in numbers, it's useful to write the present value of the Hubble expansion rate, as it's often written, as 100 times h sub 0 kilometers per second per megaparsec. So this defines h sub 0, little h sub 0.

And I think the main advantage of using this notation is that you don't have to keep writing kilometers per second per megaparsec which gets to be a real pain to keep writing. So little h sub 0 is just a dimensionless number that defines the Hubble expansion rate.

And it does then allow you to write other formulas in simple ways. Numerically, Newton's constant you can look up. It's 6.672 times 10 to the minus 8 centimeter cubed per gram per second squared. And when you put these equations together, all you need to know is G and H squared to know what rho critical is.

You find that rho critical can be written initially for any H0 as 1.88 h0 squared coming from the h squared in the original formula times 10 to the minus 29 grams per centimeter cubed. And note the whopping smallness of that 10 to the minus 29.

The mass density of our universe is, as far as we know, equal to this critical density. We know it's equal to within about a half of a percent or so. And h0 is near 1. h0, according to Planck, is 0.67-- according to the Planck satellite measurement of the Hubble parameter

And if you put that into here, you get the critical density is about 8.4 times 10 to the minus 30 grams per centimeter cubed. And that is equivalent to about 5 protons per cubic meter.

So I've written the answer in terms of grams per cubic centimeter because to me that's a very natural unit for density because it's the density of water. We're saying that the average density of the universe is only about 10 to the minus 29 quantity 8.410. The average density of the universe is only about 10 to the minus 29 times the density of water. So it's an unbelievably empty universe that we're living in. It's hard to believe the universe is that empty, but there are large spaces between the galaxies that we look at. So the universe is incredibly empty.

And in fact, this is a vastly better vacuum. An average part of the universe is a vastly better vacuum than can be made on Earth by any machinery that we have access to. So the best vacuum is empty space, just middle of nowhere. And it's vastly better than what we can actually produce on Earth. Yes.

AUDIENCE: I see you used protons here, 5 protons per cubic medium. So is this density corresponding to density of baryonic matter, or all types of matter?

PROFESSOR: This is actually all types of matters, even though I'm using my proton as a meter stick. But it is the total mass density of the universe that's very close to the critical value. And I was just about to say something about what the total mass is made up of.

Cosmologists define-- this was certainly mentioned in my first lecture-- a Greek letter capital Omega to mean the actual mass density of the universe divided by this critical density. So omega equals 1 in this language corresponds to a flat universe at this critical point. Omega bigger than 1 corresponds to a closed universe. And omega less than 1 corresponds to an open universe.

And today, we know that omega is equal to 1 to an accuracy of about a half of a percent. To a very good accuracy we know omega is very close to 1. It's made up of different contributions. And these tend to vary with time-- as the best measurements tend to vary with time by a few percent.

But omega matter-- and here I mean visible plus dark matter-- is roughly about 0.3. And most of the universe today, as we mentioned earlier, the universe today is pretty much dark energy-dominated. So omega dark energy is about equal to 0.70. And omega total is pretty close to 1. Plus or minus about a half of a percent.

So one of the implications here is that we've been assuming in our calculation so far that we're talking about nothing but non-relativistic matter. That's actually only about 30% of the actual matter in the current universe. So I did say this is the beginning.

The current universe today does not obey the equations that we've written down very accurately. It's pretty far off. But the equations that we wrote down are pretty accurate for the history of our universe from a period of about 50,000 years after the Big Bang up to about 9 billion years after the Big Bang. Yes.

AUDIENCE: Before dark energy was discovered, did they think omega was [INAUDIBLE]?

PROFESSOR: Yes. At least many people did. Before dark energy was discovered, there was a controversy in the community over what we thought omega was. Those of us who had faith in inflation believed that omega would be 1 because that's what inflation predicts.

Astronomers who just had faith in observations believed that omega was 0.2 or 0.3 because that's what they saw. And the truth ended up being somewhat in between in the sense that omega total we now all agree is very close to 1 as inflation predicts. But it's still true that the stuff that the astronomers saw at this earlier time did only add up to 0.2 or 0.3. So they correctly estimated what they were looking at and they had no way of knowing that there was also this dark energy component until it was finally discovered in 1998. Yes.

AUDIENCE: If we don't know what dark energy is, observationally how have we been able to measure that?

PROFESSOR: How do we measure it so accurately if we don't know what it is, right? Right.

Well, the answer is while we're not sure what it is, we actually do think we know a lot of its properties. And essentially, almost all properties that are relevant to cosmology. We just don't know what's sort of like inside. So we know it creates repulsive gravity. We know how much repulsive gravity it creates. And we also know to reasonable accuracy how the dark energy has been evolving with time, which is really that it's not been evolving with time. And that determines what its pressure is.

It determines, in fact-- to not evolve with time we'll see later requires the pressure to be equal to the negative of the energy density. Pressure is related to how energies change with time, as I mentioned a few minutes ago in a different context.

If you have a box that expands and there's a pressure, the pressure does dp dv work on the box. And you can tell how much the energy in the box should change for a given pressure. And we'll do this more carefully later, but to have the energy not change at all requires a pressure, which is the negative of the energy density.

So we know how much acceleration the dark energy causes. We know to reasonable accuracy and we assume it's true that the pressure is equal to minus the energy density. And that's all you need to know to calculate how much of it you need to account for that much acceleration. And that's how it's done. Any other questions?

OK next thing we want to do is to actually solve this equation for the easiest case. We'll solve it in general later, but the easiest case to solve it is the case of the critical case.

We only have a few minutes, but it only takes a few minutes to solve the equation for this case. It should be over here.

So for the critical case, it's the case E is equal to 0. And therefore, we just have a dot squared is equal to a constant divided by a. And it won't really matter for us right now what this constant is. So I don't even have to keep track of it. I'll just write it as const, C-O-N-S-T.

And I'll take the square root of this equation because it's easier to know what a dot is than to know what a is. Easier to make use of knowing what a dot is. So I can rewrite that equation as a dot, which I'll now write as da dt, to be a little more explicit about what we're talking about, is equal to a constant over a to the 1/2. So this now is the k equals 0 evolution. So this is just the same Friedmann equation rewritten for the special case k equals 0.

And now I'm just going to perform the amazingly complicated manipulation of multiplying both sides of the equation by a to the half. So we have a to the half. I'm also going to multiply by dt. So a to the half da will be equal to a constant times dt.

And now we can just integrate both sides as an indefinite integral. And integrating both sides as an indefinite integral, the left-hand side becomes-- go back over here. 2/3 a to the 3/2 is equal to a constant times t plus an arbitrary other constant of integration. This is the most general equation, which when differentiated gives you this.

And now, this equation can be solved to tell us what a is as a function of t. But before I do that, I'm going to say something about c prime here. c prime is allowed by the integration. But remember that when we defined our scale of t, we just started at some arbitrary time, ti, which we didn't even specify. So there's no particular significance to the origin of time in the equations that we've written so far. So we're perfectly free to shift the origin of time by just redefining our clocks.

Cosmic time, remember, is defined in a way which makes it uniform throughout the universe by our construction. But we haven't said anything yet about how to start cosmic time. But now, we have a good way to start it.

In this model, there is going to be a time at which a is going to go to 0. No matter what we choose for c prime here, there will be some t which will make the right-hand side 0. And therefore, a 0. And that's the instant of the Big Bang. That's when everything starts. a never gets smaller than 0. So it's very natural to take that to be defined to be the 0 of time. So that's what we're going to do.

So we're going to define t equals 0 to be when a of t equals 0. That's just a choice of the origin of time, which you're certainly allowed to do without contradicting anything else that we've said.

And that means we're just setting c prime equal to 0. So that when that's 0, that's 0. So this implies c prime equals 0. And that then implies-- we could take the 2/3 power of this equation. And I told you we don't really care what that constant is. So therefore, we don't really care about what that constant is.

What we get is that a is equal to some constant, not necessarily related to any of the constants we've said so far. Although, you can calculate how it is. But some constant times t to the 2/3 power. Or equivalently, you could just write a is proportional to t to the 2/3, which has the same content.

Now, you might think you'd want to know what the constant of proportionality is. But remember, the constant of proportionality just depends on the definition of the notch. If you want to define the notch so that a is equal to 1 today, then you would care what the constant is. If you're willing to just leave the definition of the notch arbitrary, then you don't care what the constant is.

And that's the case that I'll be doing, actually. I will not define a to be 1 today. The definition of the notch is just arbitrary as far as the equations that I'll be writing. And therefore, it will be sufficient to know that a is just proportional to t to the 2/3 for the flat universe case, for the critical density case.

And that's where we'll stop today. And this actually pretty really covers everything through lecture notes three, which is the same material that will be covered on the quiz next week.

[INAUDIBLE] does not seem to have shown up, but we'll assume that probably the review session will be next Monday night at 7:30. I will check it out and get back to you by email.