

8.286 Lecture 14  
October 29, 2013

# THE GEODESIC EQUATION

## Summary of Lecture 13: Adding Time to the Robertson-Walker Metric

$$ds^2 = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} .$$

Meaning:

- ★ If  $ds^2 > 0$ , it is the square of the spatial separation measured by a local free-falling observer for whom the two events happen at the same time.
- ★ If  $ds^2 < 0$ , it is  $-c^2$  times the square of the time separation measured by a local free-falling observer for whom the two events happen at the same location.
- ★ If  $ds^2 = 0$ , then the two events can be joined by a light pulse.

## Adding Time to the Robertson-Walker Metric

$$ds^2 = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} .$$

Application: **The Geodesic Equation.**

## THE GEODESIC EQUATION

Notation:  $ds^2 = g_{ij}(x^k) dx^i dx^j$  .

Description of path from  $A$  to  $B$ :

$x^i(\lambda)$ , where  $x^i(0) = x^i_A$ ,  $x^i(\lambda_f) = x^i_B$  .

Distance from  $x^i(\lambda)$  to  $x^i(\lambda + d\lambda)$ :

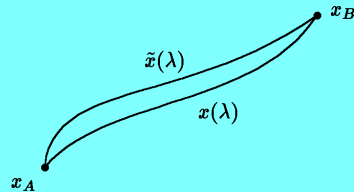
$$ds^2 = g_{ij}(x^k(\lambda)) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} d\lambda^2 .$$

Distance from  $A$  to  $B$ :

$$S[x^i(\lambda)] = \int_0^{\lambda_f} \sqrt{g_{ij}(x^k(\lambda)) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}} d\lambda .$$

Varying the path (calculus of variations, 1696):

$$\begin{aligned} \tilde{x}^i(\lambda) &= x^i(\lambda) + \alpha w^i(\lambda) , \\ w^i(0) &= 0 , \quad w^i(\lambda_f) = 0 , \\ \left. \frac{dS[\tilde{x}^i(\lambda)]}{d\alpha} \right|_{\alpha=0} &= 0 \\ &\text{for all } w^i(\lambda) , \end{aligned}$$



where

$$S[\tilde{x}^i(\lambda)] = \int_0^{\lambda_f} \sqrt{A(\lambda, \alpha)} d\lambda , \quad A(\lambda, \alpha) = g_{ij}(\tilde{x}^k(\lambda)) \frac{d\tilde{x}^i}{d\lambda} \frac{d\tilde{x}^j}{d\lambda} .$$

$$\begin{aligned} \left. \frac{dS[\tilde{x}^i(\lambda)]}{d\alpha} \right|_{\alpha=0} &= \frac{1}{2} \int_0^{\lambda_f} \frac{1}{\sqrt{A(\lambda, 0)}} \\ &\times \left\{ \frac{\partial g_{ij}}{\partial x^k} w^k \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} + 2g_{ij} \frac{dw^i}{d\lambda} \frac{dx^j}{d\lambda} \right\} d\lambda . \end{aligned}$$

**Integrate second term by parts!**

$$\left. \frac{dS}{d\alpha} \right|_{\alpha=0} = \int_0^{\lambda_f} \left\{ \frac{1}{2\sqrt{A}} \frac{\partial g_{jk}}{\partial x^i} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} - \frac{d}{d\lambda} \left[ \frac{1}{\sqrt{A}} g_{ij} \frac{dx^j}{d\lambda} \right] \right\} w^i(\lambda) d\lambda .$$

$$\Rightarrow \frac{d}{d\lambda} \left[ \frac{1}{\sqrt{A}} g_{ij} \frac{dx^j}{d\lambda} \right] = \frac{1}{2\sqrt{A}} \frac{\partial g_{jk}}{\partial x^i} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} .$$

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