8.286 Lecture 10
October 10, 2013

INTRODUCTION TO NON-EUCLIDEAN SPACES

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(After finishing dynamics of homogeneous expansion)

Summary of Lecture 9

Evolution of a closed universe:

First order Friedmann equation $\Longrightarrow$

$$\int_0^{\tilde{t}_f} d\tilde{t} = \int_0^{\tilde{a}_f} \frac{\tilde{a}\, d\tilde{a}}{\sqrt{2\alpha\tilde{a} - \tilde{a}^2}} \,, \quad \text{where } \tilde{a} \equiv \frac{a}{\sqrt{k}} \,, \quad \tilde{t} \equiv ct \,,$$

$$\text{and} \qquad \alpha \equiv \frac{4\pi}{3}\frac{G\rho\tilde{a}^3}{c^2} \,.$$

Substitute $\tilde{a} - \alpha = -\alpha\cos\theta \quad \Longrightarrow$

$$\begin{aligned} \tilde{t}_f &= \alpha(\theta_f - \sin\theta_f) \\ \tilde{a}_f &= \alpha(1 - \cos\theta_f) \end{aligned} \quad \Longrightarrow \quad \boxed{\begin{aligned} ct &= \alpha(\theta - \sin\theta) \\ \frac{a}{\sqrt{k}} &= \alpha(1 - \cos\theta) \end{aligned}}$$

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Massachusetts Institute of Technology
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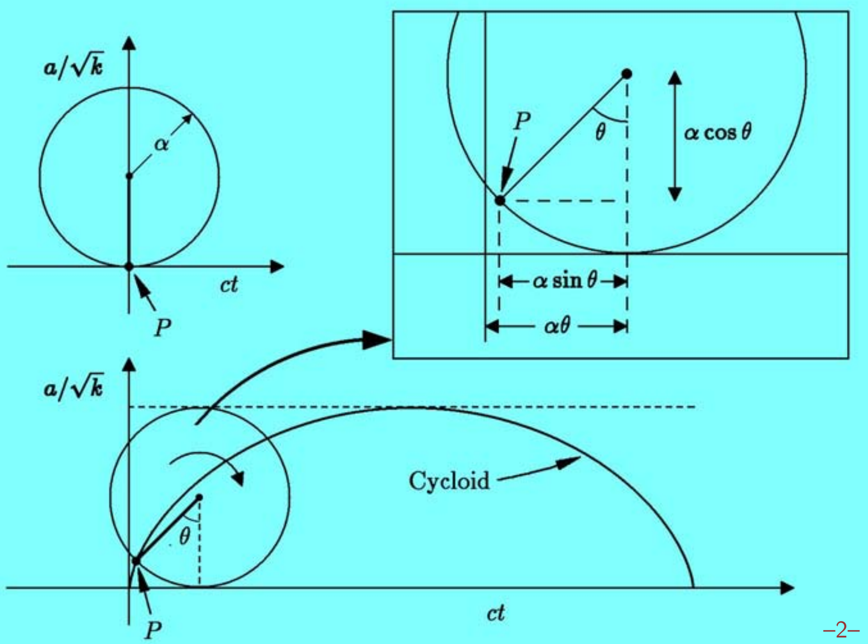


Age of a closed universe: (Want it in terms of H and Ω)

$$H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \quad \Longrightarrow \quad \tilde{a} = \frac{a}{\sqrt{k}} = \frac{c}{|H|\sqrt{\Omega - 1}} .$$

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho\tilde{a}^3}{c^2} \quad \Longrightarrow \quad \alpha = \frac{c}{2|H|}\frac{\Omega}{(\Omega-1)^{3/2}} .$$

$$\frac{a}{\sqrt{k}} = \alpha(1-\cos\theta) \quad \Longrightarrow \quad \frac{c}{|H|\sqrt{\Omega-1}} = \frac{c}{2|H|}\frac{\Omega}{(\Omega-1)^{3/2}}(1-\cos\theta).$$

$$\Longrightarrow \quad \sin\theta = \pm\frac{\sqrt{\Omega-1}}{\Omega} .$$

Then $ct = \alpha(\theta - \sin\theta) \quad \Longrightarrow$

$$t = \frac{\Omega}{2|H|(\Omega-1)^{3/2}}\left\{\arcsin\left(\pm\frac{2\sqrt{\Omega-1}}{\Omega}\right) \mp \frac{2\sqrt{\Omega-1}}{\Omega}\right\} .$$

$$t = \frac{\Omega}{2|H|(\Omega-1)^{3/2}}\left\{\arcsin\left(\pm\frac{2\sqrt{\Omega-1}}{\Omega}\right) \mp \frac{2\sqrt{\Omega-1}}{\Omega}\right\} .$$

Quadrant	Phase	Ω	Sign Choice	$\theta = \sin^{-1}()$
1	Expanding	1 to 2	Upper	0 to $\frac{\pi}{2}$
2	Expanding	2 to ∞	Upper	$\frac{\pi}{2}$ to π
3	Contracting	∞ to 2	Lower	π to $\frac{3\pi}{2}$
4	Contracting	2 to 1	Lower	$\frac{3\pi}{2}$ to 2π

Evolution of an Open Universe

The calculations are almost identical, except that one defines

$$\tilde{a} \equiv \frac{a}{\sqrt{\kappa}} , \qquad \text{where} \qquad \kappa \equiv -k > 0 .$$

One finds hypergeometric functions instead of trigonometric functions, with

$$ct = \alpha(\sinh\theta - \theta)$$
$$\frac{a}{\sqrt{\kappa}} = \alpha(\cosh\theta - 1)$$

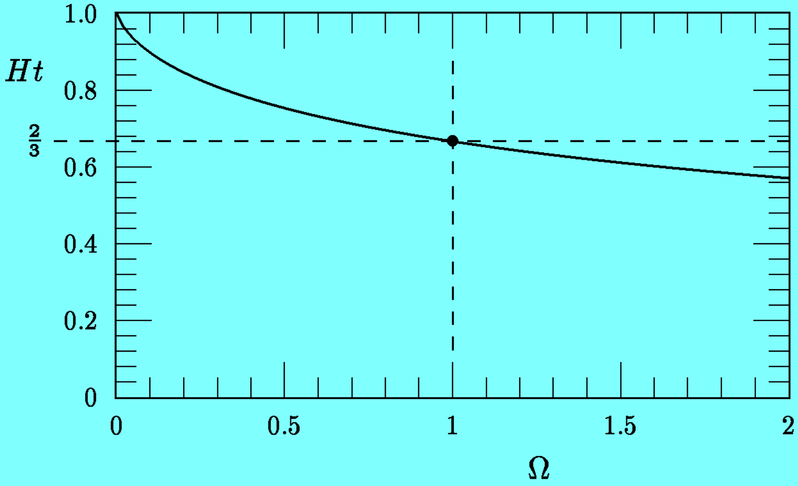
instead of

$$ct = \alpha(\theta - \sin\theta)$$
$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) .$$

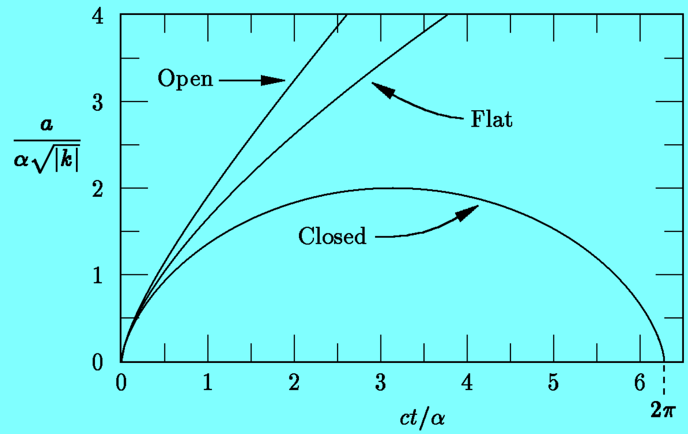
The Age of a Matter-Dominated Universe

$$|H|\,t = \begin{cases} \dfrac{\Omega}{2(1-\Omega)^{3/2}}\left[\dfrac{2\sqrt{1-\Omega}}{\Omega} - \sinh^{-1}\left(\dfrac{2\sqrt{1-\Omega}}{\Omega}\right)\right] & \text{if } \Omega < 1 \\ 2/3 & \text{if } \Omega = 1 \\ \dfrac{\Omega}{2(\Omega-1)^{3/2}}\left[\sin^{-1}\left(\pm\dfrac{2\sqrt{\Omega-1}}{\Omega}\right) \mp \dfrac{2\sqrt{\Omega-1}}{\Omega}\right] & \text{if } \Omega > 1 \end{cases}$$

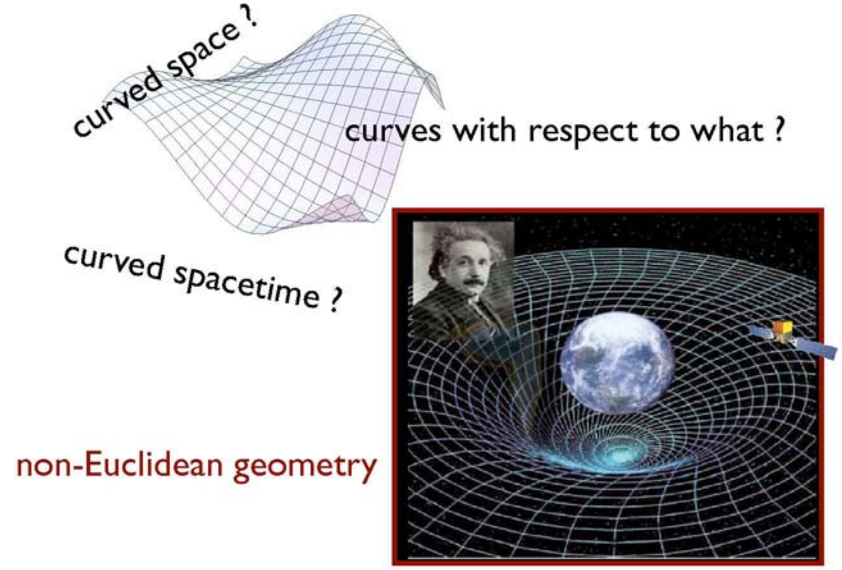
Age of Matter-Dominated Universe



Evolution of a Matter-Dominated Universe



INTRODUCTION TO NON-EUCLIDEAN SPACES



Note on image: I have added the "Dali" clock and Fermi satellite images to the original image created for the Gravity Probe B collaboration

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Euclid's *Postulates*

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given a straight line segment , a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which the angles are less than two right angles

Corrected 10/10/13

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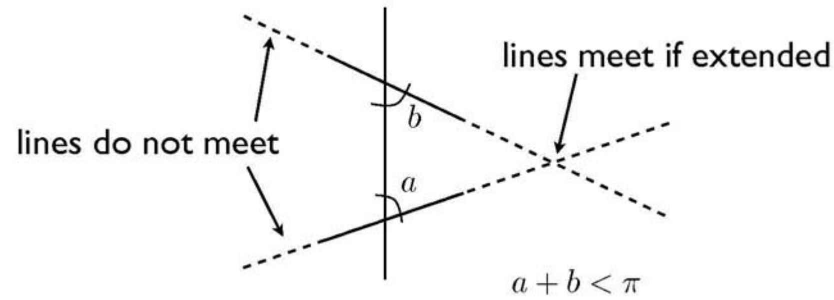
5th Postulate



5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which the angles are less than two right angles

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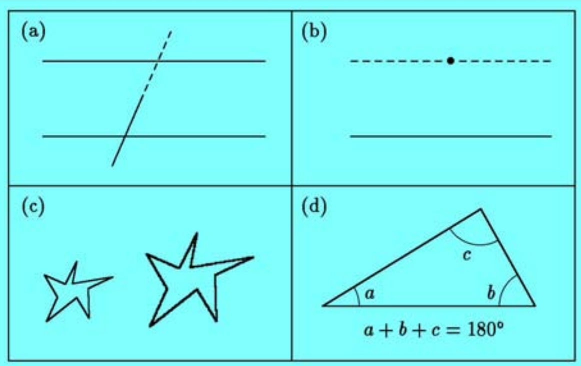
Equivalent Statements of the 5th Postulate



(a) "If a straight line intersects one of two parallels (i.e, lines which do not intersect however far they are extended), it will intersect the other also."

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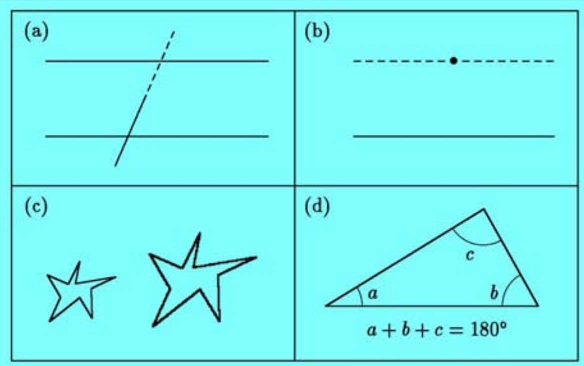
Equivalent Statements of the 5th Postulate



(b) "There is one and only one line that passes through any given point and is parallel to a given line."

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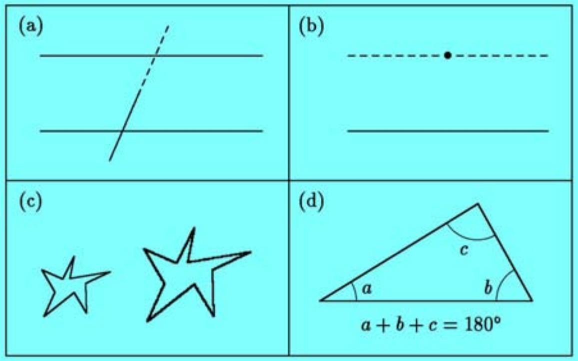
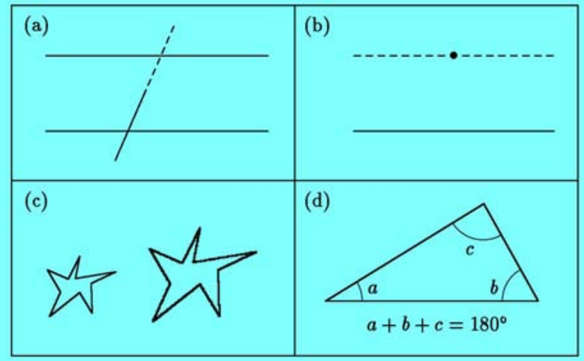
Equivalent Statements of the 5th Postulate



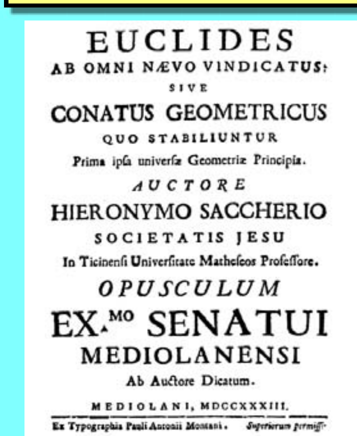
(c) "Given any figure there exists a figure, similar to it, of any size." (Two polygons are similar if their corresponding angles are equal, and their corresponding sides are proportional.)

Equivalent Statements of the 5th Postulate



(d) "There is a triangle in which the sum of the three angles is equal to two right angles (i.e., 180°)."

Giovanni Geralamo Saccheri (1667–1733)


EUCLIDES
AB OMNI NÆVO VINDICATUS:
SIVE
CONATUS GEOMETRICUS
QUO STABILIUNTUR
Prima ipsa universæ Geometriæ Principia.
AUCTORE
HIERONYMO SACCHERIO
SOCIETATIS JESU
In Ticinensi Universitate Matheseos Professore.
OPUSCULUM
EX.MO SENATUI
MEDIOLANENSI
Ab Auctore Dicatum.
MEDIOLANI, MDCCXXXIII.
Ex Typographia Pauli Antonii Montani. Superiorum permissu.

In 1733, Saccheri, a Jesuit priest, published *Euclides ab omni naevo vindicatus (Euclid Freed of Every Flaw)*.

The book was a study of what geometry would be like if the 5th postulate were false.

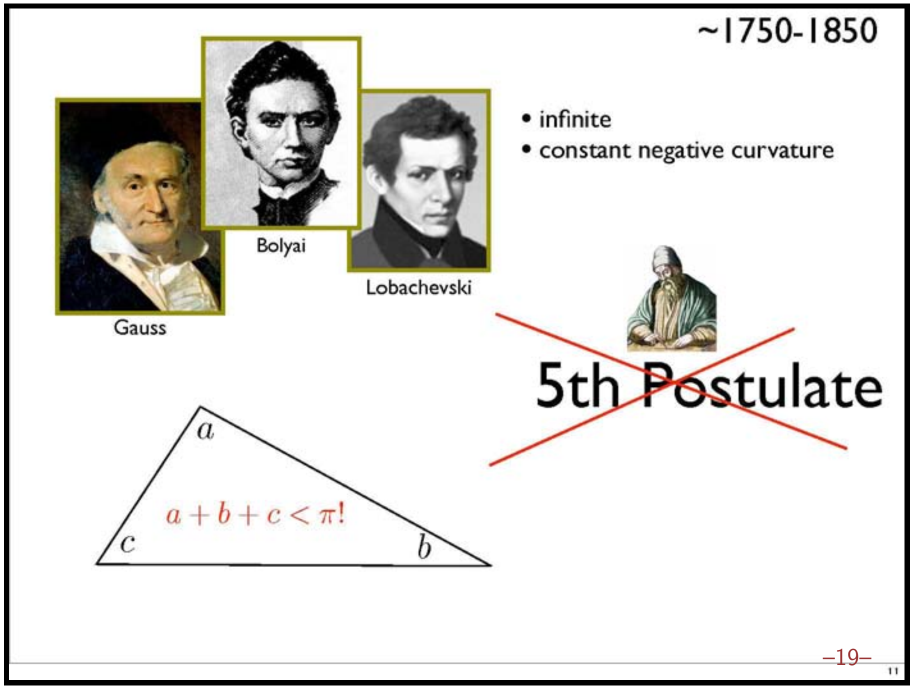
He hoped to find an inconsistency, but failed.

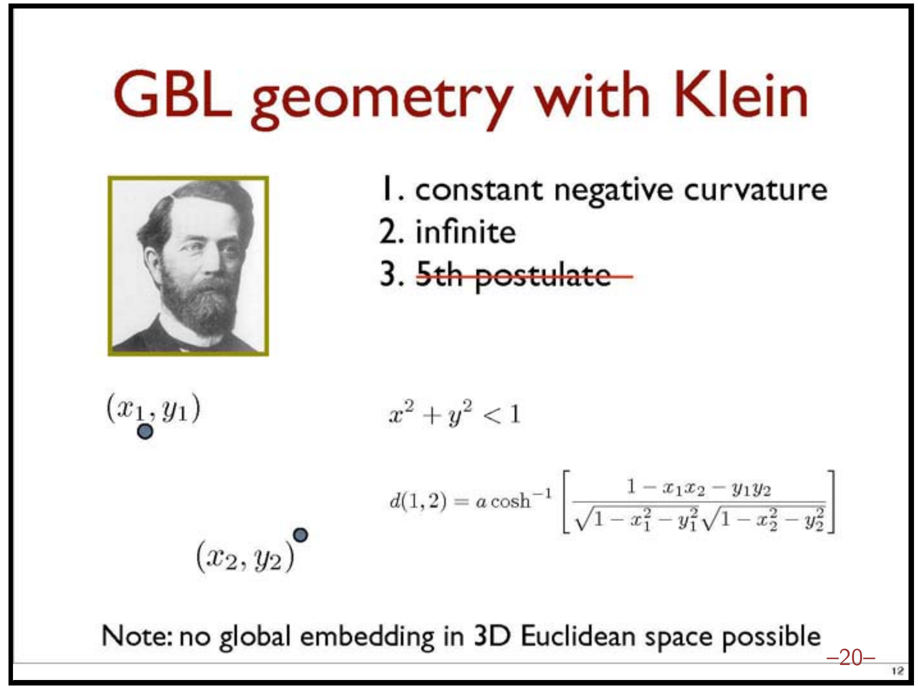
Carl Friedrich Gauss (1777–1855)

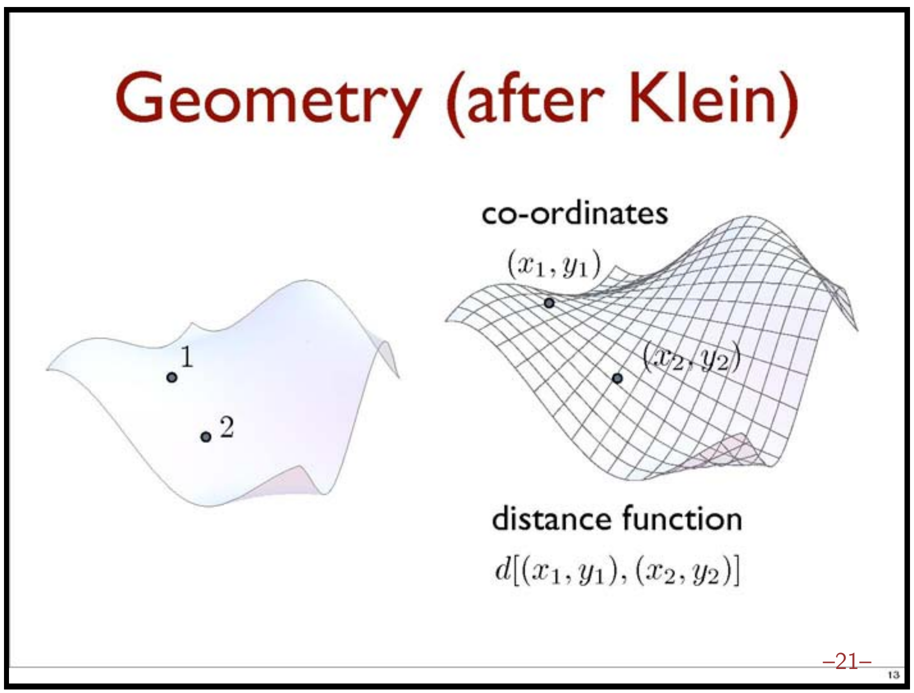
German mathematician and physicist.

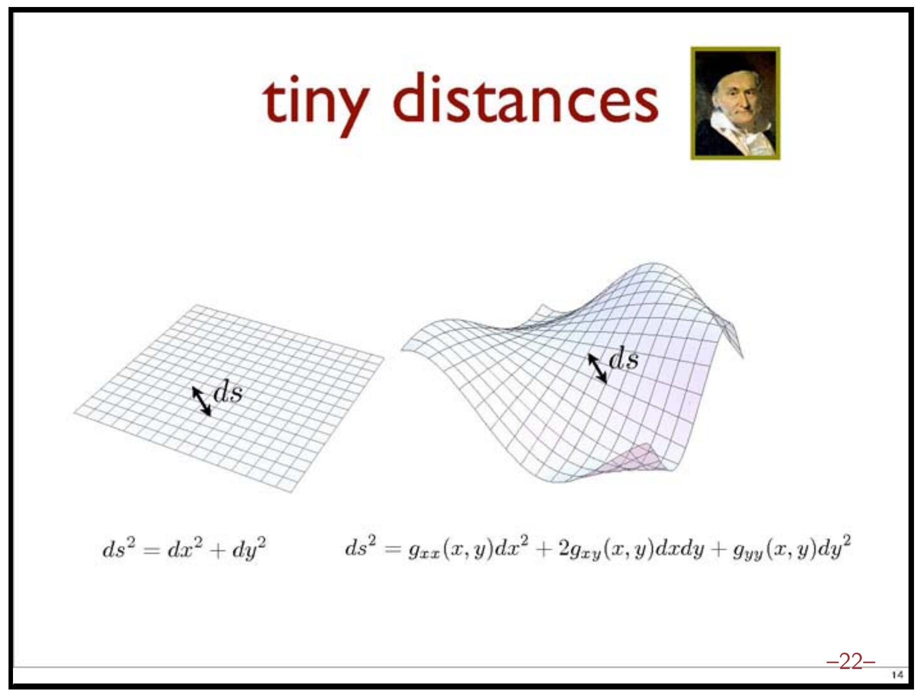
Born as the son of a poor working-class parents. His mother was illiterate and never even recorded the date of his birth.

His students included Richard Dedekind, Bernhard Riemann, Peter Gustav Lejeune Dirichlet, Gustav Kirchhoff, and August Ferinand Möbius.

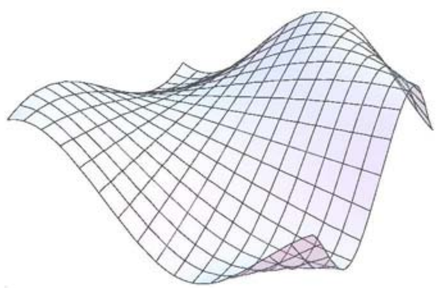

~1750-1850
Gauss
Bolyai
Lobachevski
• infinite
• constant negative curvature
~~5th Postulate~~
a
b
c
$a + b + c < \pi!$
–19–
11


GBL geometry with Klein
1. constant negative curvature
2. infinite
3. ~~5th postulate~~
(x_1, y_1)
(x_2, y_2)
$x^2 + y^2 < 1$
$d(1,2) = a \cosh^{-1}\left[\frac{1 - x_1x_2 - y_1y_2}{\sqrt{1-x_1^2-y_1^2}\sqrt{1-x_2^2-y_2^2}}\right]$
Note: no global embedding in 3D Euclidean space possible
–20–
12


Geometry (after Klein)
1
2
co-ordinates
(x_1, y_1)
(x_2, y_2)
distance function
$d[(x_1, y_1), (x_2, y_2)]$
–21–
13

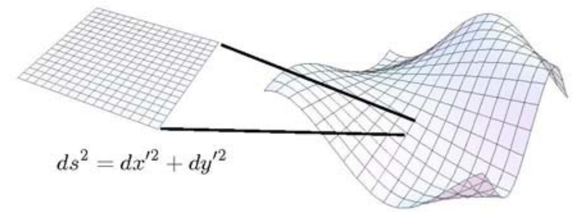

tiny distances
ds
ds
$ds^2 = dx^2 + dy^2$
$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$
–22–
14

quadratic form



$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

locally Euclidean



$$ds^2 = g_{xx}(x,y)dx^2 + 2g_{xy}(x,y)dxdy + g_{yy}(x,y)dy^2$$

$$g_{xx}g_{yy} - g_{xy}^2 > 0$$