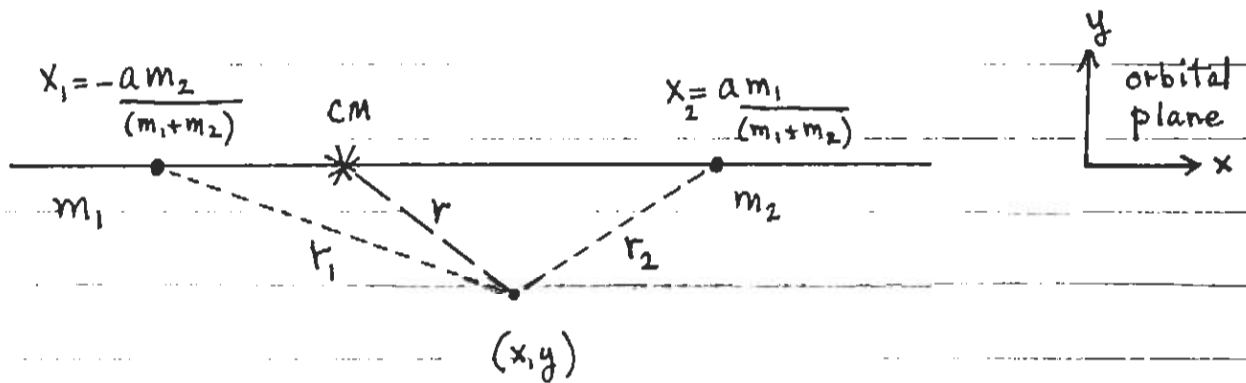


The Roche Potential

The Roche potential is defined as the effective potential of two point gravitational masses described in a frame of reference that rotates with the orbital period (taken to be circular). This potential, which has great application to binary stellar systems, can be directly understood with 8.01-level physics. Consider two point masses m_1 and m_2 located along the x -axis at $x_1 = -a m_2 / (m_1 + m_2)$ and $x_2 = a m_1 / (m_1 + m_2)$.



This configuration yields a binary separation equal to a and a center of mass located at $x=0$. The masses remain fixed in this diagram since the coordinate system is taken to be rotating at the orbital frequency.

Let us now compute the potential at an arbitrary point (x, y, z) . (Potential is usually defined as the work required to move a unit mass from infinity to a specified point.) In the absence of rotation, the potential would be given by:

$$\psi(x, y, z) = -\frac{Gm_1}{r_1} - \frac{Gm_2}{r_2}.$$

In the rotating frame, however, there is a fictitious force per unit mass, \mathcal{F}_c (the centrifugal force), that is of the form

$$\mathcal{F}_c = \Omega^2 r = \Omega^2 (x^2 + y^2)^{1/2},$$

where Ω is the orbital frequency. Integration of this force leads to a fictitious potential, ψ_c , given by

$$\psi_c = -\frac{\Omega^2}{2} r^2 = -\frac{\Omega^2}{2} (x^2 + y^2).$$

Note that ψ_c has been arbitrarily taken to be zero at $r=0$.

(Additive constants do not affect any conclusions that are drawn from a potential diagram.)

Inclusion of this fictitious potential yields a total effective potential in the rotating frame of

$$\psi(x, y, z) = -\frac{Gm_1}{\sqrt{(x-x_1)^2 + y^2 + z^2}} - \frac{Gm_2}{\sqrt{(x-x_2)^2 + y^2 + z^2}} - \frac{\Omega^2}{2} (x^2 + y^2)$$

where z is the distance above or below the orbital plane.

This expression for ψ can be simplified by using Kepler's law,

$$\Omega^2 = \frac{G(m_1 + m_2)}{a^3},$$

to yield

$$\psi(x, y, z) = \frac{GM}{a} \left[-\frac{m_1 a}{M \sqrt{(x-x_1)^2 + y^2 + z^2}} - \frac{m_2 a}{M \sqrt{(x-x_2)^2 + y^2 + z^2}} - \frac{(x^2 + y^2)}{2} \right],$$

where $M \equiv m_1 + m_2$. Now, if we define the following dimensionless lengths: $X = x/a$, $Y = y/a$, and $Z = z/a$, we obtain the desired form of the Roche potential:

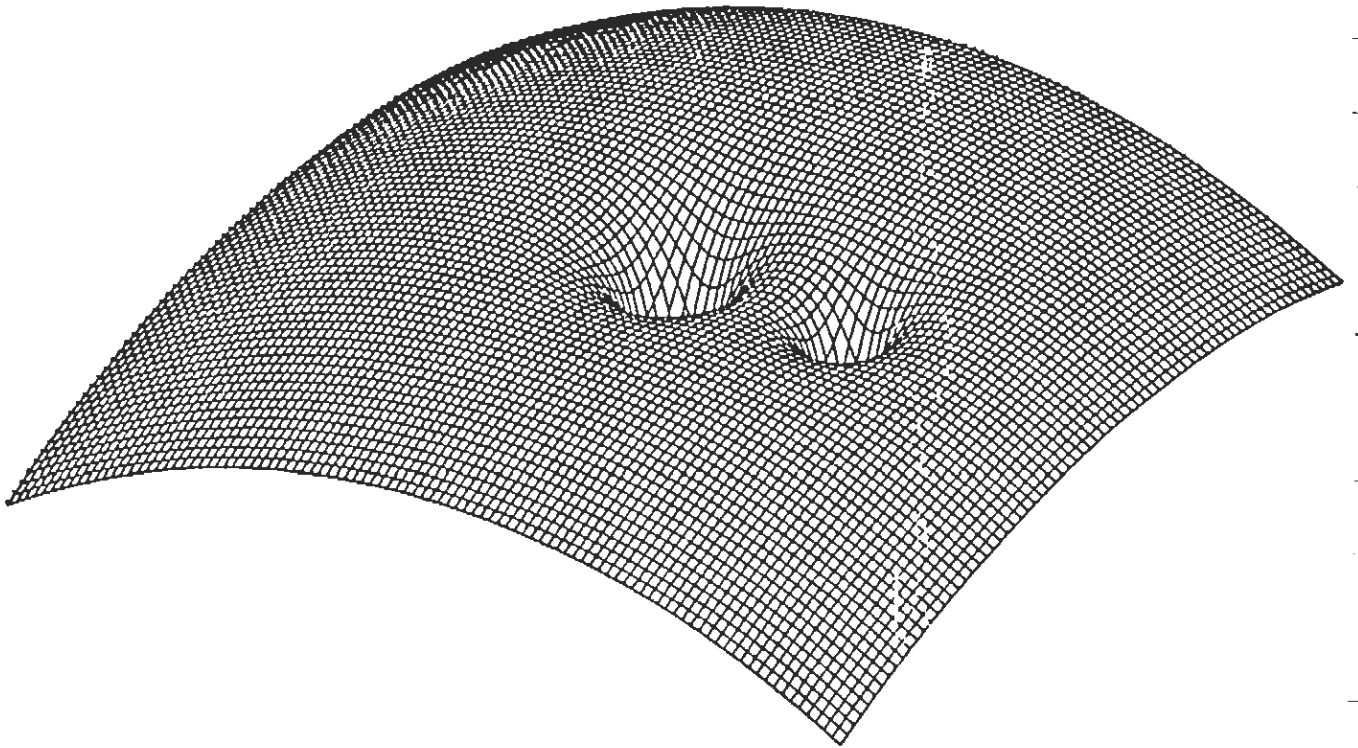
$$\psi(X, Y, Z) = \psi_0 \left[-\frac{q}{(1+q)\sqrt{(X-X_1)^2 + Y^2 + Z^2}} - \frac{1}{(1+q)\sqrt{(X-X_2)^2 + Y^2 + Z^2}} - \frac{(X^2 + Y^2)}{2} \right]$$

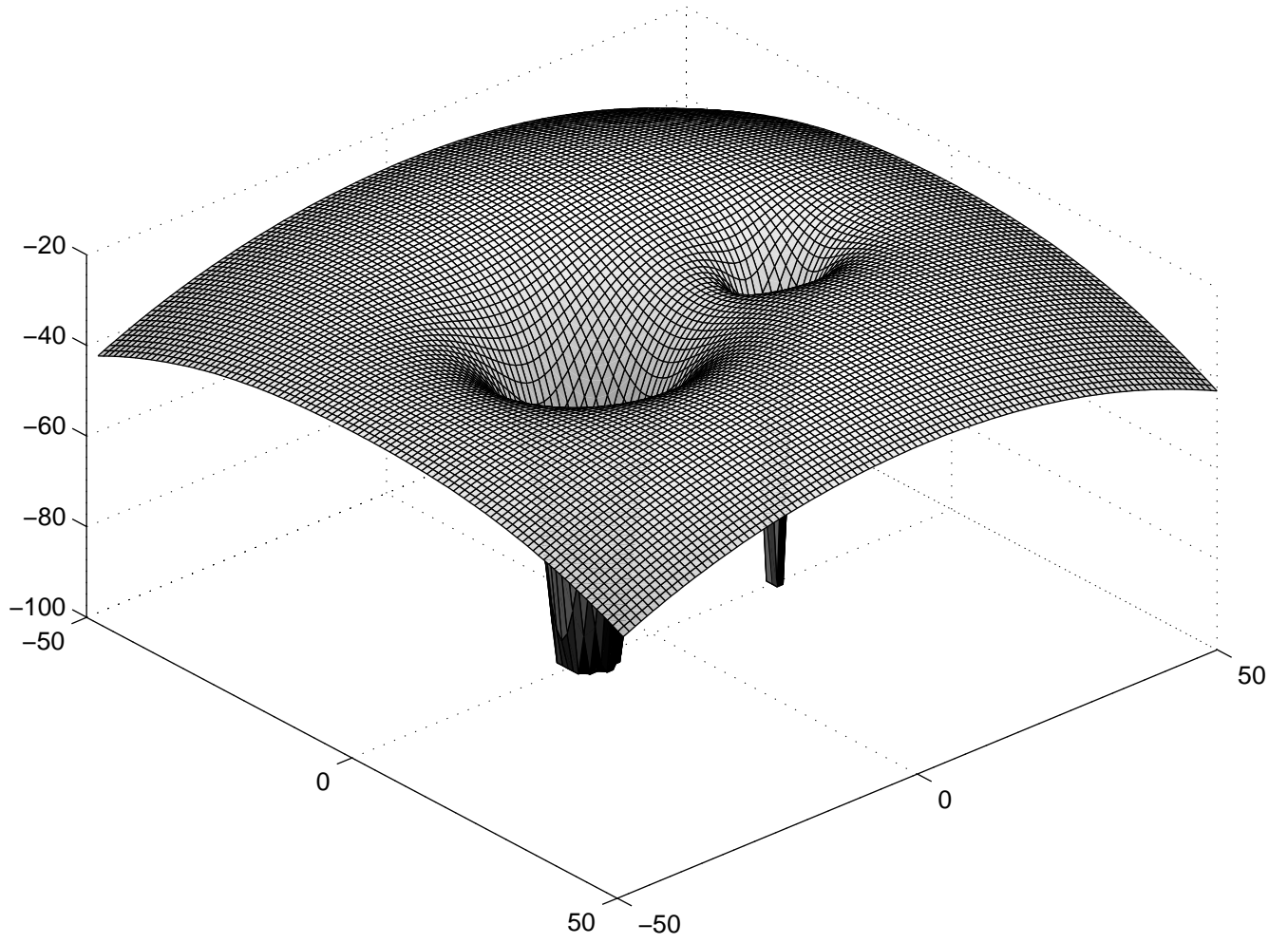
where ψ_0 is a constant associated with a particular binary system, and q is the mass ratio m_1/m_2 .

This dimensionless form of the Roche potential has the following features:

1. To within simple overall scale factors (i.e., potential in units of ψ_0 and lengths in units of the binary separation) the shape of the equipotential surfaces depends only on the mass ratio $q = m_1/m_2$.
2. Very near either mass, the potential behaves simply as $1/r$, as expected for an isolated point mass.
3. Very far from the center of mass of the binary, the dominant part of the potential is just the centrifugal repulsion term.

4. All five of Lagrange's maxima (points where there is no force on a stationary test mass) are implicitly contained in $\psi(x, y, z)$
5. Equipotential surfaces of ψ describe the approximate shapes of stars of finite size whose centers are located at x_1 and x_2 .
6. A perspective plot of ψ in the orbital plane [$\psi(x, y, z=0)$] is shown below.





$$\text{In[1]} := f = m1 / \text{Sqrt}[(x + m2)^2 + y^2] + m2 / \text{Sqrt}[(x - m1)^2 + y^2] + (x^2 + y^2) / 2$$

$$\text{Out[1]} = \frac{1}{2} (x^2 + y^2) + \frac{m2}{\sqrt{(-m1 + x)^2 + y^2}} + \frac{m1}{\sqrt{(m2 + x)^2 + y^2}}$$

$$\text{In[13]} := dx = \partial_x f$$

$$\text{Out[13]} = x - \frac{m2 (-m1 + x)}{((-m1 + x)^2 + y^2)^{3/2}} - \frac{m1 (m2 + x)}{((m2 + x)^2 + y^2)^{3/2}}$$

$$\text{In[14]} := dy = \partial_y f$$

$$\text{Out[14]} = y - \frac{m2 y}{((-m1 + x)^2 + y^2)^{3/2}} - \frac{m1 y}{((m2 + x)^2 + y^2)^{3/2}}$$

$$\text{In[15]} := dx2 = \partial_{x,x} f$$

$$\text{Out[15]} = 1 + \frac{3 m2 (-m1 + x)^2}{((-m1 + x)^2 + y^2)^{5/2}} - \frac{m2}{((-m1 + x)^2 + y^2)^{3/2}} + \frac{3 m1 (m2 + x)^2}{((m2 + x)^2 + y^2)^{5/2}} - \frac{m1}{((m2 + x)^2 + y^2)^{3/2}}$$

$$\text{In[16]} := dy2 = \partial_{y,y} f$$

$$\text{Out[16]} = 1 + \frac{3 m2 y^2}{((-m1 + x)^2 + y^2)^{5/2}} - \frac{m2}{((-m1 + x)^2 + y^2)^{3/2}} + \frac{3 m1 y^2}{((m2 + x)^2 + y^2)^{5/2}} - \frac{m1}{((m2 + x)^2 + y^2)^{3/2}}$$

$$\text{In[17]} := dx dy = \partial_{x,y} f$$

$$\text{Out[17]} = \frac{3 m2 (-m1 + x) y}{((-m1 + x)^2 + y^2)^{5/2}} + \frac{3 m1 (m2 + x) y}{((m2 + x)^2 + y^2)^{5/2}}$$

$$\text{In[23]} := \text{Simplify}[dx /. \{y \rightarrow \frac{\sqrt{3}}{2}, m1 \rightarrow 1 - m2, x \rightarrow \frac{(m1 - m2)}{2}\}]$$

$$\text{Out[23]} = 0$$

$$\text{In[24]} := \text{Simplify}[dy /. \{y \rightarrow \frac{\sqrt{3}}{2}, m1 \rightarrow 1 - m2, x \rightarrow \frac{(m1 - m2)}{2}\}]$$

$$\text{Out[24]} = 0$$

$$\text{In[25]} := \text{Simplify}[dx2 /. \{y \rightarrow \frac{\sqrt{3}}{2}, m1 \rightarrow 1 - m2, x \rightarrow \frac{(m1 - m2)}{2}\}]$$

$$\text{Out[25]} = \frac{3}{4}$$

$$\text{In[26]} := \text{Simplify}[dy2 /. \{y \rightarrow \frac{\sqrt{3}}{2}, m1 \rightarrow 1 - m2, x \rightarrow \frac{(m1 - m2)}{2}\}]$$

$$\text{Out[26]} = \frac{9}{4}$$

$$\text{In[27]} := \text{Simplify}[dx dy /. \{y \rightarrow \frac{\sqrt{3}}{2}, m1 \rightarrow 1 - m2, x \rightarrow \frac{(m1 - m2)}{2}\}]$$

$$\text{Out[27]} = -\frac{3}{4} \sqrt{3} (-1 + 2 m2)$$