MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department
Earth, Atmospheric, and Planetary Sciences Department

Astronomy 8.282J-12.402J — March 1, 2006

Problem Set 4

Due: Wednesday, March 8 (in lecture)

Reading: Conclude your reading of chapters 8 & 9 in Zeilik & Gregory. Light reading of Chapter 10 (we will not be covering "The Sun: A Model Star" in the lectures). Start Chapter 11.

Reminder: Quiz #1 will be given on Monday, March 13 during the regular lecture period. Please bring a calculator. You may also prepare and use a page of notes, formulae, facts, etc.

Problem 1

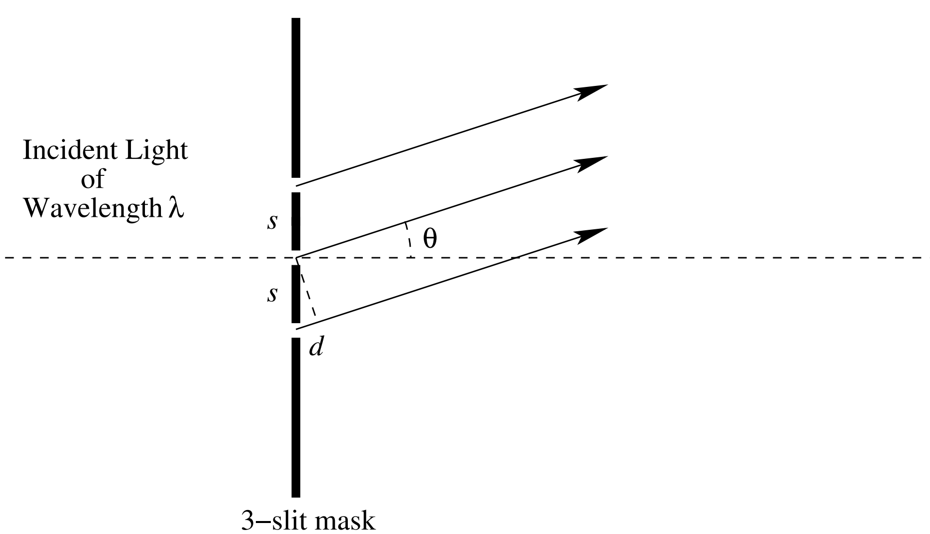
"Short Problems on the Sensitivity and Angular Resolution of Telescopes"

Zeilik & Gregory, Chapter 9, Problems: #1, #3–first part, #6a, #8a, #10 (page 198); and #12 (page 199).

Problem 2

"Three-Slit Diffraction Problem"

Following the derivation of the Fraunhofer diffraction pattern for two narrow slits, compute the diffraction pattern for three equally spaced narrow slits. (See the sketch below for details.)



Note that $d = s \sin\theta \approx s\theta$ for small θ.

Useful Relations:

$$\cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2) = 2\cos\theta_1\cos\theta_2$$
$$\cos^2\theta = \tfrac{1}{2}[1 + \cos(2\theta)]$$

Problem 3

"Divergence of a Laser Beam"

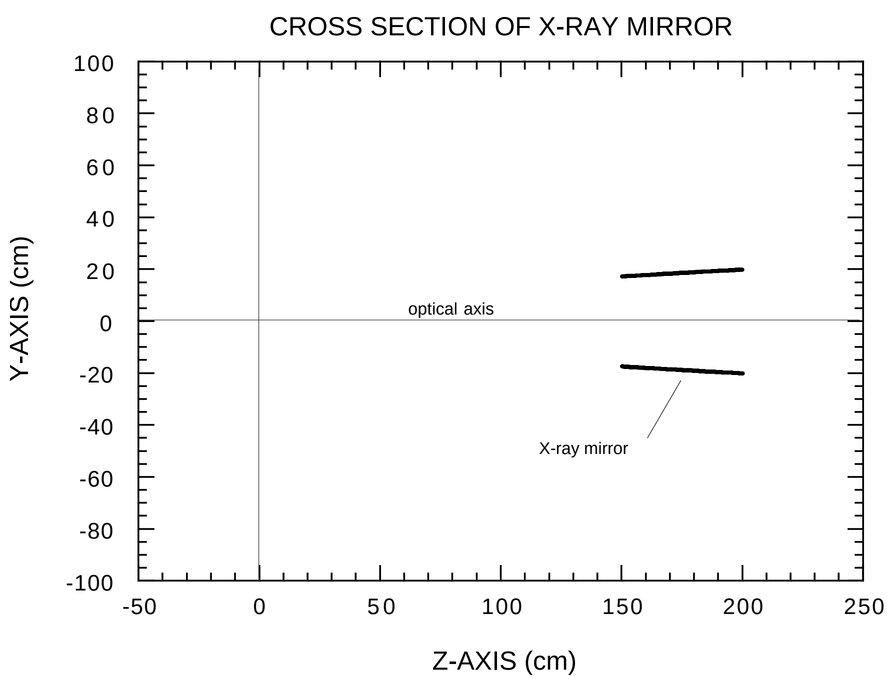
A laser beam of wavelength 6328Å has a diameter of 1.0 cm when it exits the laser. If the laser beam is aimed at the moon, find the approximate minimum diameter of the light beam when it reaches the moon. (Hint: The laser beam will diverge because of diffraction effects. The distance to the moon is 3.8×10^{10} cm.)

Problem 4

"X-Ray Mirror"

The cross section (in the $Y - Z$ plane) of a grazing incidence X-ray mirror has the shape shown in the sketch below. The reflecting surface is a paraboloid of rotation given by $z = 0.5(x^2 + y^2)$. Only the segment between $z = 150$ cm and $z = 200$ is actually used for reflecting X-rays. For X-rays incident along the optical axis of the mirror ($\hat{z}$):

a. Find the geometrical collecting area of this mirror.
b. Find the average grazing angle (i.e., complement to the angle of incidence) for an X-ray beam traveling along the $-\hat{z}$ direction.
c. Use the information in the class handout to estimate approximately the shortest wavelength of X-rays (in Å) that will be efficiently reflected by this mirror.
d. (Optional) Find the coordinate z_{focus} where X-rays will be brought to a focus.



Problem 5

"Semiclassical Derivation of the Energy Levels of the Hydrogen Atom"

Compute the quantized energy levels of the hydrogen atom using the following procedure and assumptions:

a. Find a general analytic expression for the total energy (kinetic plus potential) of an electron of mass m in a circular orbit of radius r about a proton of mass M. Take the proton to be infinitely more massive that the electron (i.e., $M \gg m$).
b. Find an expression for the orbital angular momentum of the electron in part (a).
c. Combine the results of parts (a) and (b) to eliminate the dependence on the variable r.
d. What are the allowed energy levels, E_n, of hydrogen under the hypothesis that the orbital angular momentum L is quantized, i.e., occurs only in units of $\hbar$ (Planck's constant/2π), i.e., $L = n\hbar$, where n is an integer > 0?
e. Cast your expression in the form

$$E_n = -\frac{R}{n^2},$$

where E_n is the energy of the nth level, and R is the Rydberg constant. Find the Rydberg constant in terms of fundamental constants of Nature.
f. If an electron makes a transition between levels n_1 and n_2 a quantum of light (photon) with energy $h\nu$ will be emitted. Find a general expression for the frequency of the light emitted in such a transition.
g. Compute the wavelength of $H\alpha$ radiation ($n = 3$ to $n = 2$ transition).

Problem 6 (Optional)

"Short Optics Problems"

a. Show that a ray of light that passes through a slab of glass with parallel sides emerges with only a parallel displacement.
b. Show that a ray of light that passes through a narrow wedge prism of angle β (with $\beta \ll 1$) is deflected from its original direction by an angle $\beta(n-1)$, where n is the index of refraction of the glass.

Problem 7 (Optional)

"Snell's Law Derived From Fermat's Principle"

Fermat's Principle states that the path of a light ray from one point to another is that which requires the least time. Suppose that light is emitted by a source at $(x = 0, y = y_0)$ in air, heads in the direction of a semi-infinite slab of glass located at $y = 0$, and enters the glass such as to reach the point $(x = x_0, y = -y_0)$. The geometry is shown in the diagram. Take the speed of light to be c in air, and c/n in glass, where n is called the "index of refraction", and is typically a number in the range of 1.4–1.6.

Assume, from Fermat's Principle, the obvious result that *within* any given medium (i.e., of fixed n) light travels in a straight line. Then use Fermat's Principle to prove Snell's law: $\sin\theta_i = n \sin\theta_t$.

Hint: take the distances traveled to be $y_0/\cos\theta_i$ and $y_0/\cos\theta_t$ in the air and glass, respectively. Then note that $x_0 = y_0(\tan\theta_i + \tan\theta_t)$, which defines the constraint between θ_i and θ_t. You should convince yourself that the relation $x_0 = y_0(\tan\theta_i + \tan\theta_t)$ is, in fact, correct.