Lecture 16 - Topics

- Light-cone fields and particles

Reading: Sections 9.5-10.1

Space-filling D-brane

$$\boxed{x^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma)}$$

$$\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$$
$$\alpha_{-n}^\mu = \sqrt{n}a_n^\mu$$
$$\alpha_n^\mu = \sqrt{n}a_n^\mu$$

$$n\cdot x(\tau,\sigma) = \beta\alpha'(n\cdot p)\tau$$

$n\cdot\mathcal{P}^\tau$ constant along string.

$$\ddot{x}^\mu - x^{\mu\prime\prime} = 0$$
$$\mathcal{P}_\mu^\sigma = -\frac{1}{2\pi\alpha'}\frac{\partial x^\mu}{\partial\sigma}$$
$$\mathcal{P}_\mu^\tau = \frac{1}{2\pi\alpha'}\frac{\partial x^\mu}{\partial\tau}$$
$$(\dot{x}\pm x')^2 = 0$$

$$\begin{aligned}\dot{x}^\mu(\tau,\sigma) &= \sqrt{2\alpha'}\alpha_0^\mu + \sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\mu e^{-in\tau}\cos(n\sigma)\\ &= \sqrt{2\alpha'}\sum_{n\in Z}\alpha_n^\mu e^{-in\tau}\cos(n\sigma)\end{aligned}$$

$$\begin{aligned}x'^\mu(\tau,\sigma) &= -i\sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\mu e^{-in\tau}\sin(n\sigma)\\ &= -i\sqrt{2\alpha'}\sum_{n\in Z}\alpha_n^\mu e^{-in\tau}\sin(n\sigma)\end{aligned}$$

$$\boxed{\dot{x}^\mu \pm x'^\mu = \sqrt{2\alpha'}\sum_{n\in Z}\alpha_n^\mu e^{-in(\tau\pm\sigma)}}$$

Let $\eta\mu = (1/\sqrt{2}, 1/\sqrt{2}, 0, 0, \ldots)$

$$n \cdot x = \frac{1}{\sqrt{2}}(x^0 + x^1) = x^+$$
$$n \cdot p = p^+$$

$$\boxed{x^+(\tau, \sigma) = \beta\alpha' p^+ \tau}$$

Open Strings:

$$x^+ = 2\alpha' p^+ \tau$$
$$x_0^+ = 0$$
$$\alpha_n^+ = \alpha_{-n}^+ \text{ for } n = 1, \ldots, \infty$$

Coordinates of string: (x^+, x^-, x^I) $I = 2, 3 \ldots, d$. Each x^I has associated x_0^I,α_n^I.

Remarkable statement: Can generate fulll solution of wave equation using the constraints.

Recall: $a \cdot b = -a^+b^- - a^-b^+ + a^I b^I$, $(\dot{x} \pm x')^2 0$

$$-\partial(\dot{x}^+ \pm x^{+\prime})(\dot{x}^- \pm x^{-\prime}) + (\dot{x}^I \pm x^{I\prime})^2 = 0$$

Here x^- appears linearly with number coefficient, so we can solve for x^- and get string motion.

$$(\dot{x}^- \pm x^{-\prime}) = \frac{1}{2}\frac{1}{2\alpha' p^+}(\dot{x}^I \pm x^{I\prime})^2$$

Determines for you $\dot{x}^-$ and $x^{-\prime}$. If you know $x^-(P)$ can get $x^-(Q)$ where P and Q are any two points.

$$\partial_\sigma \dot{x}^- = \partial_\tau x^{-\prime}$$

For closed strings, paths from P to Q not deformable. Think about? Is true an extra constraint?

Let's solve this finally:

$$\begin{aligned}
\dot{x}^- \pm x^{-\prime} &= \sqrt{2\alpha'} \sum_{n \in Z} \alpha_n^- e^{-in(\tau \pm \sigma)} \\
&= \frac{1}{2}\frac{1}{2\alpha'} 1p^+(2\alpha') \sum_{p,q \in Z} \alpha_p^I \alpha_q^I e^{[-i(p+q)(\tau \pm \sigma)]} \\
&= \frac{1}{p^+} \sum_{n \in Z} \frac{1}{2} \sum_{p \in Z} \alpha_p^I \alpha_{n-p}^I e^{(-in(\tau \pm \sigma))}
\end{aligned}$$

$$\boxed{\sqrt{2\alpha'}\alpha_n^- = \frac{1}{2p^+}\sum_{p\in Z}\alpha_p^I\alpha_{n-p}^I}$$

Now solved motion of open string for any constraints.

Transverse Virasora Mode

$$\boxed{L_n^\perp = \frac{1}{2}\sum_{p\in Z}\alpha_p^I\alpha_{n-p}^I}$$

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}L_n^\perp$$

All of 2D field theory based on Virosora algebra. Show up everywhere in math and physics. Here we've seen the origins.

$$\sqrt{2\alpha'}p^- = \alpha_0^-$$

$$2p^+p^- = 2p^+\frac{\alpha_0^-}{\sqrt{2\alpha'}} = \frac{2p^+}{2\alpha'}(\sqrt{2\alpha'}\alpha_0^-) = \frac{1}{\alpha'}L_0^\perp$$

$$M^2 = -p^2 = 2p^+p^- - p^Ip^I = \frac{1}{\alpha'}L_0^+ - p^Ip^I$$

$$\begin{aligned}M^2 &= \frac{1}{\alpha'}\left(\frac{1}{2}\sum_{p\in Z}\alpha_p^I\alpha_{-p}^I\right) - p^Ip^I \\ &= \frac{1}{2\alpha'}(\alpha_0^I\alpha_0^I + \sum_{n=1}^{\infty}\alpha_n^I\alpha_{-n}^I) - p^Ip^I\end{aligned}$$

$$\alpha_0^I = \sqrt{2\alpha'}p^I$$

$$\boxed{M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty} n a_n^{I^*}a_n^I}$$

If no constants of oscillation, then $M = 0$, string collapses to a point. Classical string theory solved! Most complete solution of equation of motion of string. But need other things with no mass - photons, gluons, gravitons.

Prepare ground for quantization of strings.

EM fields ⇒ "photon", spin-1 field, $A^\mu(x)$
Gravity ⇒ "graviton states", spin-2 field, $h_{\mu\nu}(x)$

Scalar fields just have $\Phi(x) \in \Re$. Simplest of all! But might not exist in nature. ("Scalar" under Lorentz symmetry)

For E&M fields:

$$(\vec{E}, \vec{B}) \Rightarrow A^\mu(x) \Rightarrow F^\mu \Rightarrow S = \frac{1}{4}\int F^2$$

This took a lot of work! Will be very easy for scalar fields $\Phi(t, \vec{x})$

$$\text{KE}_{\text{density}} = \frac{1}{2}\frac{\partial\phi}{\partial x^0}\frac{\partial\phi}{\partial x^0}$$

$$\text{PE}_{\text{density}} = \frac{1}{2}M^2\phi^2$$

$$\boxed{\mathcal{L} = \frac{1}{2}\frac{\partial\phi}{\partial x_0}\frac{\partial\phi}{\partial x_0} - \frac{1}{2}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^i} - \frac{1}{2}M^2\phi^2}$$