Lecture 10 - Topics

- Relativistic strings: Nambu-Gotto action, equations of motion and boundary conditions

Reading: Section 6.1 - 6.5

$S \propto$ world-sheet area (A)
Units of const. of proportionality
$[S] = [\int dtL] =$ Energy $\times$ time $= \frac{ML^2}{T}$
[World Sheet Area] $= L^2$
[Const. of Proportionality] $= \frac{M}{T}$

Recall for tension T_0: $[T_0] =$ [Force] $= \frac{ML}{T^2}$. $\left[\frac{T_0}{c}\right] = \frac{M}{T}$. So let T_0/c be constant of proportionality for $S \propto A$.

Nambu-Gotto Action

$$S = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}$$

where $\dot{X}^\mu = \frac{\partial X^\mu}{d\tau}$. $X' = \frac{\partial X^\mu}{d\sigma}$. The two tangents to the surface.

This defines the dynamics of the string.

World sheet metric: Let $\zeta^1 = \tau$, $\zeta^2 = \sigma$.

Distance between 2 points on the worldsheet:

$$\begin{aligned} -dx^2 &= dX^\mu dX_\mu = \eta_{\mu\nu} dX^\mu dX^\nu \\ &= \eta_{\mu\nu}\frac{\partial X^\mu}{\partial \zeta^\alpha}\frac{\partial X^\nu}{\zeta^\beta} d\zeta^\alpha d\zeta^\beta \\ &= \gamma_{\alpha\beta}(\zeta) d\zeta^\alpha d\zeta^\beta \end{aligned}$$

where $\gamma_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial \zeta^\alpha}\frac{dX^\nu}{\partial \zeta^\beta}$

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X}X' \\ \dot{X}X' & (X')^2 \end{pmatrix}$$

String world-sheet metric. All signs from the η are in the X's. So:

$$\boxed{S = -\frac{T_0}{c}\iint d\tau d\sigma\sqrt{-\gamma}}$$

where $\gamma = det(\gamma_{\alpha\beta})$. Lorentz invar., reparam invar.

$$X = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \mathcal{L}(\dot{X}^\mu, X'^\mu)$$

Lagrangian Density:

$$\mathcal{L}(\dot{X}^\mu, X'^\mu) = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}$$

$$\delta S = 0$$

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_i} d\sigma \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}\right)\left(\frac{\partial(\delta X^\mu)}{\partial \tau}\right) + \left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}\right)\left(\frac{\partial(\delta X^\mu)}{\partial \sigma}\right)\right]$$

$$\delta \dot{X}^\mu = \delta \frac{\partial X^\mu}{\partial \tau}$$

Define:

$$\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = \mathcal{P}^\tau_\mu = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}$$

$$\frac{\partial \mathcal{L}}{\partial X'^\mu} = \mathcal{P}^\sigma_\mu = -\frac{T_0}{c}\frac{(\dot{X}X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{\cdots}}$$

Things will simplify soon.

So:

$$\delta S = \iint d\tau d\sigma \left[\frac{\partial}{\partial \tau}(\delta X^\mu \mathcal{P}^\tau_\mu) + \frac{\partial}{\partial \sigma}(\delta X^\mu \mathcal{P}^\sigma_\mu) - \delta X^\mu(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma})\right]$$

$\delta S = 0$ so ...

$$\boxed{\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0}$$

For open strings:

$$\int_{\tau_i}^{\tau_f} d\tau[\delta X^0(\tau,\sigma_1)\mathcal{P}^\sigma_0(\tau,\sigma_1) - \delta X^0(\tau,0)\mathcal{P}^\sigma_0(\tau,\sigma_1) + \delta X^1(\tau,\sigma_1)\mathcal{P}_1^{\sigma)(\tau,\sigma_1) - \delta X^1(\tau,0)\mathcal{P}^\sigma_1(\tau,\sigma_1) + \ldots + \delta X^d \ldots]}$$

2D Constraints.

For most, get choice as to how the term vanishes since product of 2 terms (so either can be 0).

For $\mu \neq 0$:

1. Dirichlet BCs: $X^\mu(\tau, \sigma_*) =$constant, $\delta X^\mu(\tau, \sigma_*) = 0$ (for $\sigma_* = 0$ or σ_1)
2. Free BCs: $P^\sigma_\mu(\tau, \sigma_*) = 0$ for $\sigma_* = 0$ or σ_1.

For $\mu = 0$:

1. $\partial X^0/\partial\tau < 0$ can't have Dirichlet.
2. Free BCs. $\mathcal{P}^\sigma_0(\tau, 0) = \mathcal{P}^\tau_0(\tau, \sigma_1) = 0$.