8.251 – Homework 6
Corrected 3/17/07[1]

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Due Tuesday, March 20.

1. (15 points) *Three-dimensional motion of closed strings and cusps.*

   We considered in lecture the closed string motion described by

$$\vec{X}(t,\sigma) = \frac{1}{2}\Big(\vec{F}(u) + \vec{G}(v)\Big)\,, \quad \text{with} \quad u = ct + \sigma\,,\ v = ct - \sigma\,. \tag{1}$$

   Here $\sigma \sim \sigma + \sigma_1$, where $\sigma_1 = E/T_0$ and E is the energy of the string. We showed that

$$|\vec{F}'(u)|^2 = |\vec{G}'(v)|^2 = 1\,, \tag{2}$$

$$\vec{F}'(u+\sigma_1) = \vec{F}'(u) \quad \text{and} \quad \vec{G}'(v+\sigma_1) = \vec{G}'(v)\,. \tag{3}$$

   Equations (2) and (3) imply that $\vec{F}'(u)$ and $\vec{G}'(v)$ can be described as two independent closed parameterized paths on the surface of a unit two-sphere. We assumed that the paths intersect at $u = u_0$ and $v = v_0$

$$\vec{F}'(u_0) = \vec{G}'(v_0)\,. \tag{4}$$

   The quantities u_0 and v_0 define a time t_0 and position σ_0. We showed that at $t = t_0$, the point $\sigma = \sigma_0$ on the string moves with the speed of light in the direction of $\vec{F}'(u_0)$.

   (a) We choose a coordinate system so that the cusp generated by (4) appears at the origin: $\vec{F}(u_0) + \vec{G}(v_0) = \vec{0}$. Use the Taylor expansions of $\vec{F}(u)$ and $\vec{G}(v)$ around u_0 and v_0 to prove that for σ near σ_0,

$$\vec{X}(t_0,\sigma) = \vec{T}\,(\sigma-\sigma_0)^2 + \vec{R}\,(\sigma-\sigma_0)^3 + \dots\,, \tag{5}$$

   where the vectors $\vec{T}$ and $\vec{R}$ are given by

$$\vec{T} = \frac{1}{4}\big(\vec{F}''(u_0) + \vec{G}''(v_0)\big)\,, \quad \vec{R} = \frac{1}{12}\big(\vec{F}'''(u_0) - \vec{G}'''(v_0)\big)\,. \tag{6}$$

   Assume that the intersection of the paths on the two-sphere indicated in equation (4) is regular: the paths are not parallel at the intersection and neither $\vec{F}''(u_0)$ nor $\vec{G}''(v_0)$ vanishes. Explain why $\vec{T}$ is non-zero and orthogonal to $\vec{F}'(u_0)$. In general $\vec{R}$ does not vanish, but it may under special conditions.

   (b) One can use equation (5) to show that the cusp opens up along the direction of the vector $\vec{T}$ and is contained in the plane spanned by $\vec{T}$ and $\vec{R}$. For this, align the positive y axis along $\vec{T}$, position the x axis so that $\vec{R}$ lies on the (x,y) plane, and demonstrate that near the cusp $y \sim x^{2/3}$. In what plane does the velocity of the cusp lie?

[1] Problem 1(d) was revised.

(c) Consider the functions $\vec{F}(u)$ and $\vec{G}(v)$ given by

$$\vec{F}(u) = \frac{\sigma_1}{2\pi}\left(\sin\frac{2\pi u}{\sigma_1}\,, -\cos\frac{2\pi u}{\sigma_1}, 0\right), \quad \vec{G}(v) = \frac{\sigma_1}{4\pi}\left(\sin\frac{4\pi v}{\sigma_1}\,, 0\,, -\cos\frac{4\pi v}{\sigma_1}\right). \qquad (7)$$

Verify that the conditions in (2) and (3) are satisfied. For the cusp at $t = \sigma = 0$ give its direction, the plane it lies on, and its velocity. Draw a sketch.

(d) Show that the motion of the closed string has period $\sigma_1/(4c)$. How many cusps are formed during a period? (Hint: recall that you found in Problem (7.3) an example of a situation in which the string returns to its original position in less time than the function $\vec{F}(ct+\sigma)$ takes to repeat itself. In fact, any free closed string, when viewed in its rest frame, will return to its original position in time $\sigma_1/(2c)$, where σ_1 is the period of the functions $\vec{F}'$ and $\vec{G}'$.)

2. (5 points) *Gravitational lensing by a cosmic string*

   A cosmic string produces a conical deficit angle Δ. An observer is a distance d from the cosmic string and a quasar is a distance ℓ from the cosmic string. The position of the quasar is such that the observer sees a double image, separated by an angle $\delta\phi$. Calculate $\delta\phi$ in terms of Δ, d, and ℓ, in the approximation that Δ is small.

3. (10 points) Problem 7.5.

4. (5 points) Problem 8.1.

5. (10 points) Problem 8.3.

6. (10 points) Problem 8.5.

I recommend Problem 8.2 as good practice to reinforce concepts.