# 1 Part 1: Analysis

Here are several parameters you will need (see figure 2):

| | |
|---|---|
| $m_1$ | mass of counter-weight |
| $m_2$ | mass of projectile |
| $m_b$ | mass of beam |
| $I_b$ | moment of inertia of beam **about its center of mass** |
| $l_1$ | distance from pivot to counter-weight |
| $l_2$ | distance from pivot to sling attachment |
| $l_s$ | length of sling |
| $\phi_{b0}$ | starting beam angle ($\sim 3\pi/4$) |
| $\phi_{s0}$ | starting sling angle ($\sim \pi/2$) |

To keep things simple, assume that the sling is massless and that it is always under tension so that its length is constant. Also, to avoid unnecessary complexity, treat the counter-weight and the projectile as point masses, and assume that both masses are released from rest at $t = 0$.

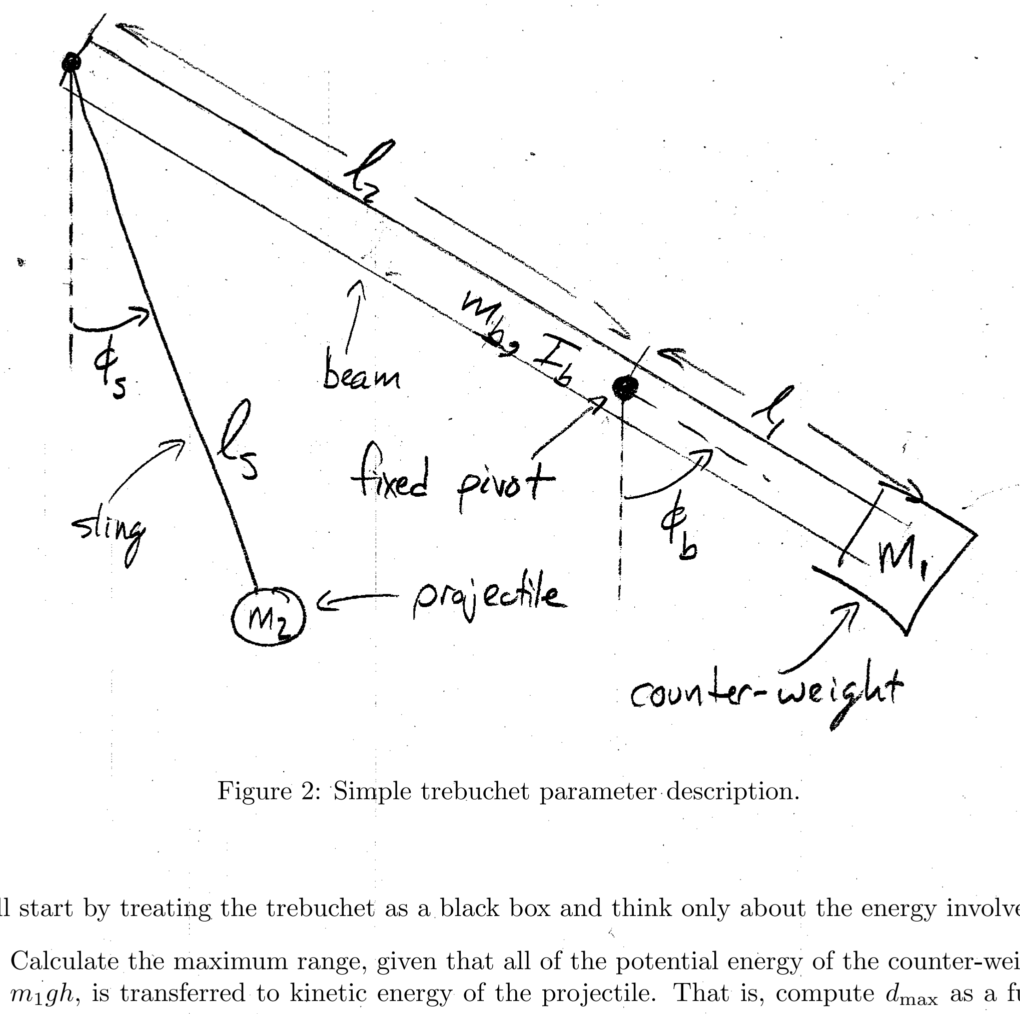


Figure 2: Simple trebuchet parameter description.

1. We'll start by treating the trebuchet as a black box and think only about the energy involved.

   (a) Calculate the maximum range, given that all of the potential energy of the counter-weight, $U_0 \approx m_1 g h$, is transferred to kinetic energy of the projectile. That is, compute $d_{\max}$ as a function of $m_1$, $m_2$, $g$, and $h$. For this calculation, treat the machine as a black-box, and consider only energy conservation. (Ignore the potential energy of the projectile, air resistance and the launch height of the projectile.)

(b) Given that all of the available potential energy of the counter-weight is transferred to the projectile, what are $\phi_b$ and $\dot{\phi}_b$ at the moment the projectile is released?

(c) To do the maximum range calculation more correctly, you should include the mass of the beam in the potential energy calculation (i.e., if the beam is long and heavy, with $l_2 \gg l_1$, it will require a lot of energy just to lift the CoM of the beam). Furthermore, the distance the counter-weight falls before reaching the bottom of the swing at $\phi_b = 0$,

$$h = y_1(t=0) + l_1 = l_1(1 - \cos\phi_{b0})$$

depends on its starting angle $\phi_b(t=0) = \phi_{b0}$. Compute the available potential energy $U_0$, and from it $d_{\max}$, given the parameters of the trebuchet in the table in part 1, assuming an initial beam angle $\phi_{b0}$ (starting from rest). (As before, ignore the potential energy of the projectile, air resistance and the launch height of the projectile.)

2. Write the kinetic and potential energy ($T$ and $U$) for the trebuchet in terms of the Cartesian coordinates of the counter-weight ($m_1$) and the projectile (or "payload", $m_2$), and the beam angle ($\phi_b$), along with whatever constants you need (e.g., lengths, masses, gravity, etc.).

3. Write the Lagrangian for the trebuchet in terms of only the angular coordinates $\phi_b$ and $\phi_s$ and constants (i.e., without individual cartesian coordinates).

4. Compute the equations of motion for $\phi_b$ and $\phi_s$ and simplify them to the form

$$\ddot{\phi}_b = \alpha_b f(\phi_b - \phi_s, \dot{\phi}_s, \ddot{\phi}_s) - \omega_b^2 \sin\phi_b \quad (1)$$
$$\ddot{\phi}_s = \alpha_s f(\phi_s - \phi_b, \dot{\phi}_b, \ddot{\phi}_b) - \omega_s^2 \sin\phi_s \quad (2)$$

to find the 3 argument function $f(a,b,c)$ and the 4 constants $\alpha_{\{b,s\}}$ and $\omega_{\{b,s\}}$.

(These equation of motion are hard to solve analytically, so I'm not asking you to do that here.)

5. Solve the equations of motion for $l_2 = 0$ to find $\phi_b(t)$ and $\phi_s(t)$ for small values of $\phi_b$ and $\phi_s$. (This is not a functional trebuchet, but it is something you can solve.)