Lecture 8: Kepler's laws

Last time: 1D potentials, reduced mass, central potentials

This time: Kepler and the $\dfrac{1}{r}$ central potential

In the early 1600s, 70 years before $F = ma$, Kepler published 3 rules followed by the planets in their orbits around the sun – Kepler's laws.

Kepler's laws:
1. Orbits are elliptical with the Sun at one focus
2. The line from the planet to the Sun sweeps out equal area in equal time
3. The square of the period of the orbit is proportional to the cube of the semimajor axis of the orbit.

Today we will see how each of these comes about, in light of Lagrangian mechanics and yesterday's lecture on central potentials.

Kepler's second law doesn't actually depend on the potential being $\frac{1}{r}$, so let's start there....
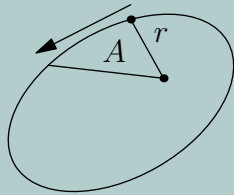
Kepler's second law says that $\frac{dA}{dt}$ is constant, and

$$\frac{dA}{dt} = \frac{1}{2}r\left(\vec{v} \cdot \hat{\phi}\right) = \frac{1}{2}r^2\dot{\phi} = \frac{L_z}{2\mu} = \text{constant}.$$

Conservation of $\vec{L}$ (angular momentum) implies Kepler's second law.

Now let's look at the effective potential for orbits under gravity. Note that the reduced mass μ is very close to the mass of the planet.
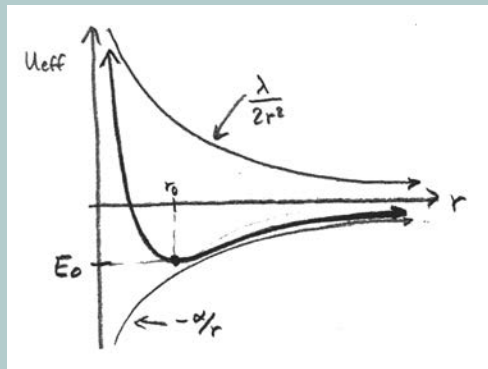
Reduced mass: $\mu = \dfrac{m_p m_s}{m_p + m_s} \approx \dfrac{m_p m_s}{m_s} = m_p$ for $m_s \gg m_p$

Gravity:

$$U(r) = -\frac{Gm_p m_s}{r} = \frac{-\alpha}{r} \implies U_{\text{eff}} = \frac{-\alpha}{r} + \frac{L_z^2}{2\mu r^2} = \frac{-\alpha}{r} + \frac{\lambda}{2r^2}$$

note: $m_p m_s = \mu M$, $\alpha = G\mu M$, $\lambda = L_z^2/\mu$

As before, we pick $\hat{z}$ orthogonal to the plane of the orbit and parallel to $\vec{L}$, to keep things easy.

$E > 0$ unbound (hyperbola)

$E = 0$ unbound (parabola)

$E_0 < E < 0$ bound (ellipse)

$E = E_0$ bound (circle)

For an orbit with a given angular momentum, we can find the minimum total energy E_0, which produces a circular orbit since $\ddot{r} = \dot{r} = 0$.

$$\frac{\partial U_{\text{eff}}}{\partial r}\bigg|_{r_0} = 0 \implies \frac{\alpha}{r_0^2} - \frac{\lambda}{r_0^3} = 0 \implies r_0 = \frac{\lambda}{\alpha} \quad \text{(radius of circular orbit)}.$$

$$\implies E_0 = \frac{1}{2}\mu \dot{r}^2 + \frac{\lambda}{2r_0^2} - \frac{\alpha}{r_0} = \frac{\alpha^2}{2\lambda} - \frac{\alpha^2}{\lambda} = \frac{\alpha^2}{2\lambda}\left(\frac{1}{2} - 1\right) = -\frac{\alpha^2}{2\lambda} = \frac{-\mu\alpha^2}{2L_z^2},$$

or $E_0 = \dfrac{-\mu}{2}\left(\dfrac{GM\mu}{L_z}\right)^2$.

(The $\frac{1}{2} - 1$ term checks out with the virial theorem: $2\bar{T} = -\bar{U}$.)

For the general case with $E \geq E_0$, we can find the EoM using conservation of energy:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{\lambda}{2r^2} - \frac{\alpha}{r} \implies \dot{r}^2 = \frac{2}{\mu}\left(E + \frac{\alpha}{r} - \frac{\lambda}{2r^2}\right)$$

$$\implies \dot{r} = \sqrt{\frac{2}{\mu}\left(E + \frac{\alpha}{r} - \frac{\lambda}{2r^2}\right)} = \frac{dr}{dt}.$$

Although this is first order, it's hard to solve for $r(t)$. But, not to worry, Landau has a trick....

$$L_z = \mu r^2 \dot{\phi} \implies \frac{d\phi}{dt} = \frac{L_z}{\mu r^2} \implies d\phi = \frac{L_z}{\mu r^2}dt$$

$$\implies \phi = \phi_0 + \int_0^t \frac{L_z}{\mu r^2}dt = \phi_0 + \int_0^r \frac{L_z}{\mu r^2}\frac{dr}{\sqrt{\frac{2}{\mu}\left(E + \frac{\alpha}{r} - \frac{\lambda}{2r^2}\right)}}.$$

Not pretty, but there is no more t, and we can do this integral to find $\phi(r)$, which describes the **shape** of the orbit.

$$\cos(\phi - \phi_0) = \left(\frac{L_z}{r} - \frac{\mu\alpha}{L_z}\right)\frac{1}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L_z^2}}}$$

$$= \sqrt{\frac{L_z^2}{2\mu\alpha^2}}\left(\frac{\alpha}{r} - \frac{\mu\alpha^2}{L_z^2}\right)\frac{1}{\sqrt{E + \frac{\mu\alpha^2}{2L_z^2}}}$$

$$= \frac{1}{2\sqrt{-E_0}}\left(\frac{\alpha}{r} + 2E_0\right)\frac{1}{\sqrt{E - E_0}}$$

$$\cos(\phi - \phi_0) = \left(1 + \frac{\alpha}{2E_0 r}\right)\frac{1}{\sqrt{1 - E/E_0}} \qquad \text{with } \frac{E}{E_0} < 1$$
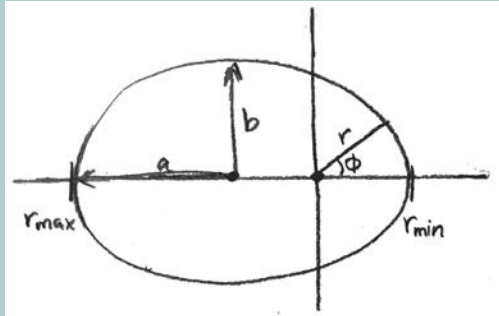
...all of which is about 1 line in LL!

We arrive at a description of the orbit, with

$$e \cos{(\phi - \phi_0)} = 1 - \frac{r_0}{r} \quad \text{with} \quad e = \sqrt{1 - \frac{E}{E_0}}, \ r_0 = -\frac{\alpha}{2E_0} = \frac{\lambda}{\alpha}$$

$$-1 \leq \cos{(\phi - \phi_0)} \leq 1 \implies r_{\max} = \frac{r_0}{1 - e} \text{ and } r_{\min} = \frac{r_0}{1 + e}.$$

$$\frac{r_0}{r} = 1 + e \cos{\phi} \text{ (with } \phi_0 = \pi \text{ such that } \phi_0 \implies r = r_{\min})$$
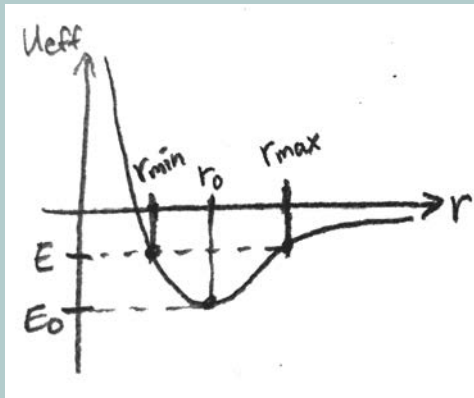

Ellipse!

$$a = \frac{r_0}{1 - e^2} = \frac{-\alpha}{2E}$$

$$b = \frac{r_0}{\sqrt{1 - e^2}} = \frac{L_z}{\sqrt{-2\mu E}}$$

Looking back to U_{eff}:

Note: $E = E_0 \implies e = 0$
and $r_{\min} = r_{\max} = r_0$

We have shown Kepler's first law, but we have dodged the third by removing time from our solution!

We can, however, use Kepler's first law and angular momentum to show that

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$$\frac{L_z}{2\mu} = \frac{dA}{dt} \implies \int_0^\tau \frac{L_z}{2\mu}\, dt = \int_0^\tau \frac{dA}{dt}\, dt \quad \text{with } \tau = \text{period}$$

$$\implies \frac{L_z}{2\mu} \cdot \tau = A = \pi ab \text{ (area of ellipse)}$$

$$\text{(using } b = \frac{L_z}{\sqrt{-2\mu E}} \text{ from above)} \implies \tau = 2\pi a^{3/2}\sqrt{\frac{\mu}{\alpha}} = \pi a \sqrt{\frac{2\mu}{-E}}$$

So the period of the orbit depends only on the planet's mass and total energy. We could have shown Kepler's third law in a totally different manner: mechanical similarity!

recall: $a = \alpha a_0$, $\tau = \beta \tau_0$, $\beta = \alpha^{1-k/2}$ for $U(r) = cr^k$.

$k = -1$ (our potential) $\implies \frac{\tau}{\tau_0} = \left(\frac{a}{a_0}\right)^{3/2} \implies \tau^2 = \frac{\tau_0^2}{a_0^3} a^3$.

Since the third law only claims "proportional to," values for τ_0 and a_0 can come from any normal orbit.

What about unbound "orbits" (trajectories)?

Unbound "orbits" or trajectories

$$E \geq 0 \implies e = \sqrt{1 - \frac{E}{E_0}} \geq 1 \text{ (recall } E_0 < 0)$$

$$r_{\min} = \frac{r_0}{1 + e}, \quad r_{\max} = \infty, \quad r_0 = \frac{L_z^2}{\mu\alpha}$$

With $U = -\dfrac{\alpha}{r}$, recall...
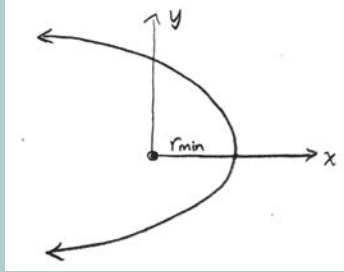
$$E < 0 \implies \text{elliptical orbits}$$
$$E = 0 \implies \text{parabolic orbits}$$
$$E > 0 \implies \text{hyperbolic orbits}$$

Let's see if this works out given our path equation:

$E = 0 \implies e = 1 \implies$ parabolic orbit? We have

$$\frac{r_0}{r} = 1 + \cos\phi \implies r\left(1 + \frac{x}{r}\right) = r_0 \implies r = r_0 - x$$

$$\implies x^2 + y^2 = r^2 = r_0^2 - 2r_0 x + x^2 \implies x = \frac{r_0^2 - y^2}{2r_0}$$

$r_{\min}$ at $y = 0 \implies r_{\min} = \frac{1}{2}r_0$

Maximum value of ϕ?

$\phi_\infty = \phi$ as $r \to \infty$
$\implies 0 = 1 + \cos\phi_\infty \implies \phi_\infty = \pm\pi$

(Note that, for hyperbolic orbits, the range of the angle ϕ is restricted by $\phi_\infty = \arccos\left(-\frac{1}{e}\right)$, e.g. $\pm\frac{2\pi}{3}$ for $e = 2$.)

So if parabolic orbits have $E = 0$, why don't they hit the origin? What makes one different from another?

$$r_0 = \frac{L_z^2}{\mu\alpha}, \quad L_z = \vec{r} \times \vec{p} = r\left(mr\dot{\phi}\right) = mrv_\perp$$

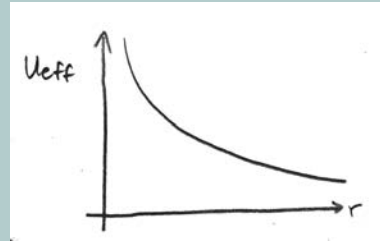$$\text{for given } L_z \text{ we have } v_\perp = \frac{L_z}{mr} \to 0 \text{ as } r \to \infty$$

(Infinity is a tricky place!)

Finally, as a lead-in to next lecture, let's look at repulsive potentials with $U = \frac{\alpha}{r}$. The math is very similar, so I'll just present the solution ...

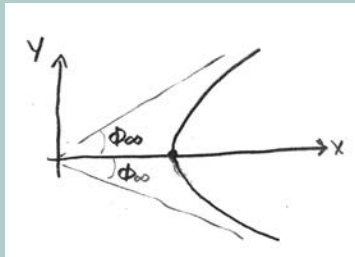for $U = \dfrac{\alpha}{r} \implies U_{\text{eff}} = \dfrac{\alpha}{r} + \dfrac{\lambda}{2r^2}$

$\alpha = GM\mu, \quad \lambda = \dfrac{L_z^2}{\mu}$

$U > 0 \implies E > 0 \implies$
unbound hyperbolic

$\dfrac{r_0}{r} = e\cos\phi - 1$ trajectories

with $r_0 = \dfrac{\lambda}{\alpha}$, we have $e = \sqrt{1 + 2E \cdot \dfrac{\lambda}{\alpha^2}} > 1$ since $E > 0$

$$r = \dfrac{r_0}{e\cos\phi - 1}, \quad r_{\min} = \dfrac{r_0}{e - 1}$$

$$\phi_\infty = \arccos\left(\dfrac{1}{e}\right)$$

Laplace-Runge-Lenz Vector

As previously discussed, a particle in 3D can have at most 5 conserved quantities $(2 \cdot 3 - 1)$. A $\frac{1}{r}$ potential has the maximum number (related to the fact that it has closed orbits). We know that E and $\vec{L}$ are conserved, but that is only 4 quantities; one is missing.

Q: Can you think of a feature of the orbits in this potential which is constant, but not determined by E or $\vec{L}$?

Example: e? No! $e = \sqrt{1 + 2E\dfrac{L_z^2}{\mu\alpha^2}}$.

A: The *phase* of perihelion (or "direction," but we know if it lies in a plane $\perp$ to $\vec{L}$).

$$\vec{e} = \frac{\dot{\vec{r}} \times \vec{L}}{\alpha} - \hat{r}. \text{ "ellipticity vector" or "Laplace-Runge-Lenz vector"}$$

$$|\vec{e}| = e, \quad \vec{e} \times \vec{L} = 0, \text{ points along semi-major axis towards } r_{\min}$$

$$\text{for } U(r) = -\frac{\alpha}{r}$$

The symmetry behind this is hidden, and will not be exposed in 8.223 (8.09!?) but you can think of it as the freedom to choose the orientation of the orbit, without changing E or $\vec{L}$.

For tomorrow: (1) decode LL 18-19, (2) Do problems 21-22 (i.e. finish pset 2) and LL problem 19.2

8.223 Classical Mechanics II
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