Massachusetts Institute of Technology

Physics Department

Physics 8.20 **IAP 2005**
Introduction to Special Relativity

Midterm Exam Solutions

1.

(a) 1. The laws of physics should take the same form in all inertial frames. 2. The speed of light c is the same in all inertial frames.
(b) 1. Newton's second law must hold in any inertial frame, which implies the transformation for x', y', z',and t' must be linear in x, y, z, and t. 2. By symmetry, $y' = y$ and $z' = z$. 3. The origin of Sigma' is moving to the right in Sigma with speed v. This allows us to constrain the coefficients in the linear transformation. 4. The origin of Sigma is moving to the left in Sigma' with speed v. This puts a further constraint on the coefficients in the linear transformation. 5. The speed of light is $c$in all frames: $x = ct$must transform to x'=ct'.
(c) 1. proper length: The length of an object as measured in a frame in which the object is at rest. 2. proper time: The time as measured by a clock in a frame in which the clock is at rest.
(d) (e) Please look at Figure(1).
(f) 1. "space-like separated": Two events A and B are said to be space-like separated if $|\Delta x_{AB}| > c|\Delta t_{AB}|$. 2. "time-like separated": Two events A and B are said to be time-like separated if $|\Delta x_{AB}| > c|\Delta t_{AB}|$. 3. "light-like separated": Two events A and B are said to be light-like separated if $|\Delta x_{AB}| = c|\Delta t_{AB}|$.
(g) The Michelson-Morley experiment measured the velocity of the earth through a hypothetical Aether, a medium in which light was postulated to undulate, and with respect to which light was postulated to propagate at speed c. The Michelson-Morley experiment used an interferometer to measure the difference in time taken by two paths of light traveling at right angles. Their experiment (conducted over a year's time, during which the earth, orbiting the sun at an orbital velocity $v \approx 10^{-4}c$, reverses its direction in the rest frame of the sun), was sufficiently sensitive that their null result

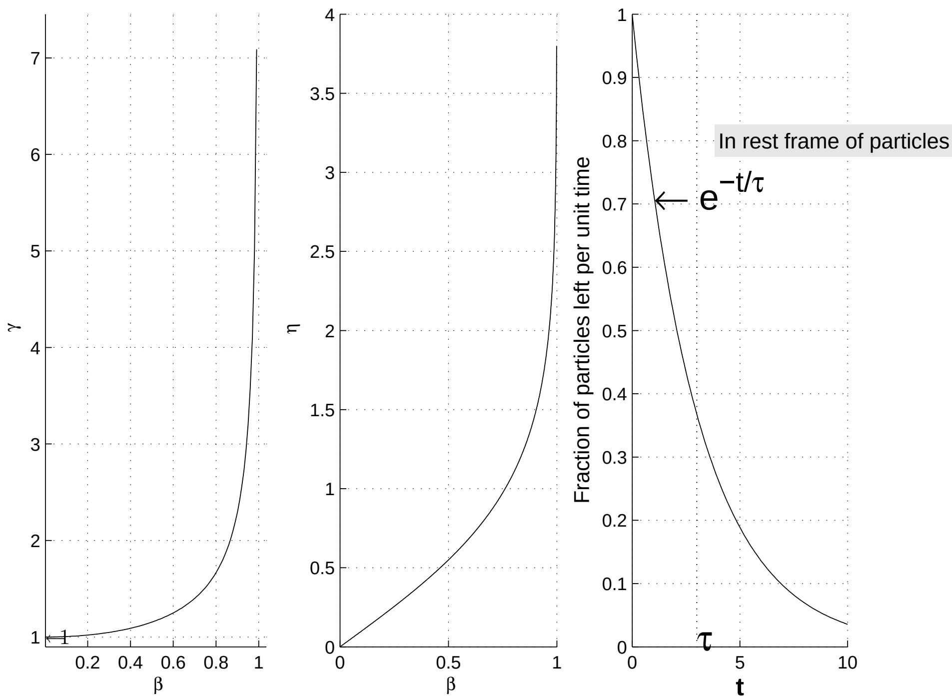


Figure 1: Problem 1(d)&(e)

implied that the earth was not in fact moving with respect to the Aether.

(h) Stellar aberration refers to the small circle that a distant star appears to trace on the sky as the earth orbits the sun during the course of one earth year. The angular radius α of this orbit is found to be given by $\tan\alpha = v/c$ under a Galilean analysis, and by $\sin\alpha = v/c$ under a (correct) Special Relativistic analysis, where v is the earth's orbital speed in the rest frame of the sun (roughly 30 km/s), and c is the speed of light. The experimental observation of this stellar aberration caused problems for the hypothesis that the Aether was dragged along with the Earth, which was one way of explaining the null result of the Michelson-Morley experiment.

(i) The Doppler shift derived in special relativity for motion parallel and anti-parallel to the signal has the same form as the Doppler shift derived in Galilean relativity, but with an additional factor of the Lorentz factor γ. Surprisingly, a Doppler shift is also found in special relativity for motion perpendicular to the signal – such a signal is redshifted by a factor of γ, while no Doppler shift is found from a Galilean analysis.

2.

(a)

$$\tan(\theta') = \frac{y'}{x'}$$
$$y' = y$$
$$x' = \frac{x}{\gamma}$$
$$\tan(\theta') = \gamma\frac{y}{x} = \gamma\tan(\theta)$$
$$\gamma = \frac{\tan(\theta')}{\tan(\theta)} = 4/3$$
$$\boxed{\beta = \frac{\sqrt{7}}{4}}$$

$$l'^2 = y'^2 + x'^2 = y^2 + \frac{x^2}{\gamma^2} = y^2 + x^2(1-\beta^2) = l_0^2 - x^2\beta^2 = l_0^2(1-\cos\theta^2\beta^2)$$

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{16}{25}$$
$$\beta^2 = 7/16$$

$$l'^2 = l_0^2(1 - \frac{7}{25}$$
$$\boxed{l' = l_0\frac{3\sqrt{2}}{5}}$$

The easy method is to compare the hypothenuse of the right triangle $(3,4,5)$ and $\left(3,3,3\sqrt{2}\right)$.

3.

`(a)` Two events has to be *time-like* separated:

$$c^2\Delta t_{AB}^2 - \Delta x_{AB}^2 > 0$$

`(b)` Simultaneous in Σ' frame if

$$\Delta t'_{AB} = 0$$

$$\Delta t'_{AB} = \gamma(\Delta t_{AB} - \beta \Delta x_{AB}/c) = 0$$

$$\boxed{\beta = \frac{c\Delta t_{AB}}{\Delta x_{AB}}}$$

`(c)` At the same point in Σ' frame if

$$\Delta x'_{AB} = 0$$

$$\Delta x'_{AB} = \gamma(\Delta x_{AB} - \beta c \Delta t_{AB}) = 0$$

$$\boxed{\beta = \frac{\Delta x_{AB}}{c\Delta t_{AB}}}$$

4.

All β's in this problem is with respect to rest (Earth) frame.

$$\texttt{approaching} : \lambda'_1 = 500 \ \texttt{nm}$$

$$\texttt{departing} : \lambda'_2 = 600 \ \texttt{nm}$$

$$\frac{\nu'_1}{\nu'_2} = \frac{\lambda'_2}{\lambda'_1} = 6/5$$

$$\nu'_1 = \sqrt{\frac{1+\beta_0}{1-\beta_0}}\nu_0 \tag{1}$$

$$\nu'_2 = \sqrt{\frac{1-\beta_0}{1+\beta_0}}\nu_0 \tag{2}$$

`(a)` Divide (1) and (2):

$$\frac{\nu'_1}{\nu'_2} = \frac{1+\beta_0}{1-\beta_0} = \frac{6}{5}$$

$$\boxed{\beta_0 = \frac{1}{11}}$$

(b) Now we want the 600 nm to appear 500 nm for the acceleration spaceship.

$$500 = \sqrt{\frac{1-\beta}{1+\beta}}600$$

$$\boxed{\beta = \frac{11}{61}}$$

Another way to solve this part is to use the concept of *relativity* : the first spaceship should be approaching the overtaking spaceship with the relative velocity of $c\beta_0$. Using the velocity transformation formula for β's we want to satisfy

$$\frac{\beta_0 - \beta}{1 - \beta_0\beta} = -\beta_0$$

$$\Rightarrow \beta = \frac{2\beta_0}{1+\beta_0^2} = \frac{11}{61}$$

(c) Now multiply both sides of (1) and (2)

$$\nu_0 = \sqrt{\nu_1'\nu_2'}$$

$$\lambda_0 = \sqrt{\lambda_1'\lambda_2'}$$

$$\boxed{\lambda_0 = 548 \text{ nm}}$$

5.

(a)

$$\boxed{\text{at time } t = \frac{d}{v}}$$

(b)

$$A : \text{noticing the asteroid}$$

$$B : \text{collision it}$$

$$\Delta \equiv {}_{\mathtt{B}} - {}_{\mathtt{A}}$$

$$\Delta x = d$$

$$\Delta t = d/v$$
$$\Delta x' = 0 \Longrightarrow \Delta t' = \text{"proper time"} = \tau$$
$$\tau = \Delta t' = \gamma(\Delta t - \beta \Delta x/c) = \gamma(d/v - \beta d/c)$$
$$= \gamma \frac{d}{v}(1 - \beta^2) = \gamma \frac{d}{v}\frac{1}{\gamma^2} = \frac{d}{v\gamma}$$
$$\boxed{\tau = \frac{d}{v\gamma}} \tag{3}$$

`(c)` Write the above equation as

$$(\gamma\tau)v = d$$

You could have written it from the beginning. The time in our frame is dilated by a factor of γ and we know she is moving with velocity v so the distance she travels in our frame is the *dilated time* times v which is d in our frame.

`(d)` Write (3) in the form:

$$v\tau = (d/\gamma)$$

This is what she will write in her frame. Since the asteroid is coming at her with velocity v and the distance is *contracted* by the factor γ and she travels for the period of τ.

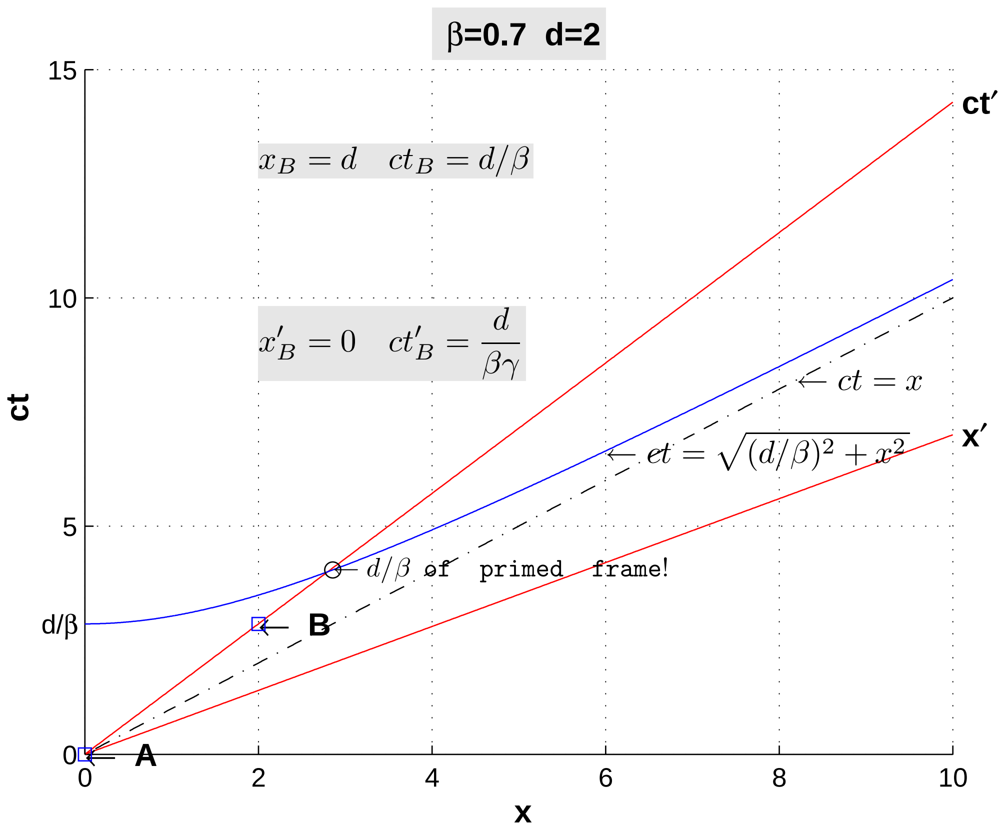
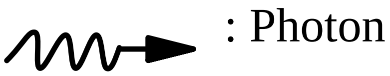


Figure 2: Problem 5

6.

(a)



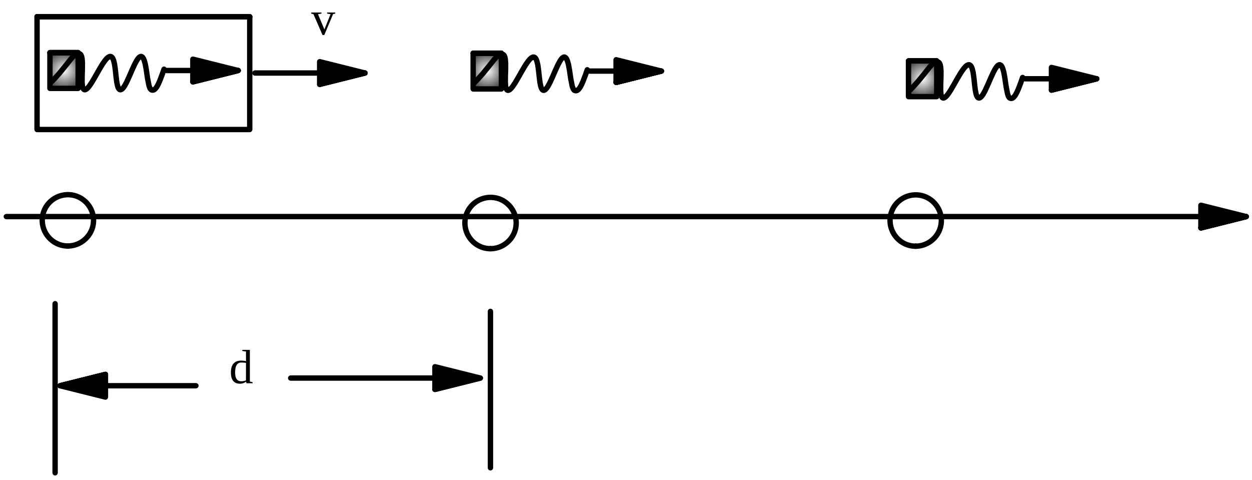


Figure 3: Problem 6

(b)

Let's denote the emission times by T
if 1 and 2 are two consecutive sites and R the distance of our observer from the first site

$$T_1 = 0$$

$$T_2 = d/v$$

$$t_1 = T_1 + R/c$$

$$t_2 = T_2 + (R - d)/c$$

$$\Delta t = t_2 - t_1 = T_2 - T_1 + (R - d)/c - R/c = d/v - d/c$$

$$\Delta t = \frac{d}{v}(1 - \beta)$$

$$\boxed{\nu = \frac{1}{\Delta t} = \frac{v}{d}\frac{1}{1 - \beta} = \nu_0 \frac{1}{1 - \beta}}$$

Another way to solve this problem (many of you tried to solve it this way) is to use the *Doppler shift* equation:

$$\nu = \sqrt{\frac{1 + \beta}{1 - \beta}}\tilde{\nu}_0$$

But $\tilde{\nu}_0$ is NOT v/d since in the frame of moving object distance d is dilated to d/γ!!

$$\tilde{\nu}_0 = \frac{v}{d/\gamma} = \frac{v\gamma}{d}$$

$$\Rightarrow \nu = \nu_0 \sqrt{\frac{1 + \beta}{1 - \beta}}\gamma = \nu_0 \sqrt{\frac{1 + \beta}{1 - \beta}}\frac{1}{\sqrt{1 - \beta^2}} = \frac{\nu_0}{1 - \beta}$$

The next step is to figure out what $\frac{1}{1-\beta}$ is *approximately* in the limit $\gamma \gg 1$:

$$\gamma \gg 1 \Longrightarrow \beta \to 1$$

$$\gamma^2 = \frac{1}{1 - \beta^2} = \frac{1}{(1 + \beta)(1 - \beta)} \simeq \frac{1}{2(1 - \beta)}$$

$$\boxed{\gamma \gg 1 \Longrightarrow \frac{1}{1 - \beta} \simeq 2\gamma^2}$$

$$\boxed{\nu \simeq \nu_0(2\gamma^2)}$$