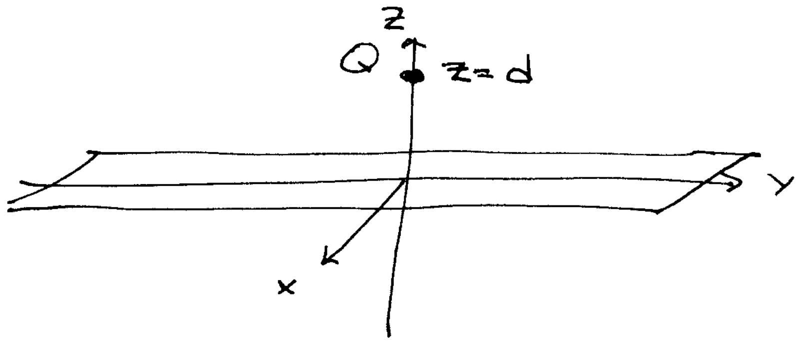
METHOD OF IMAGES

Two examples to discuss: plane + sphere
Are there other examples — sure!
(we'll get back to that)

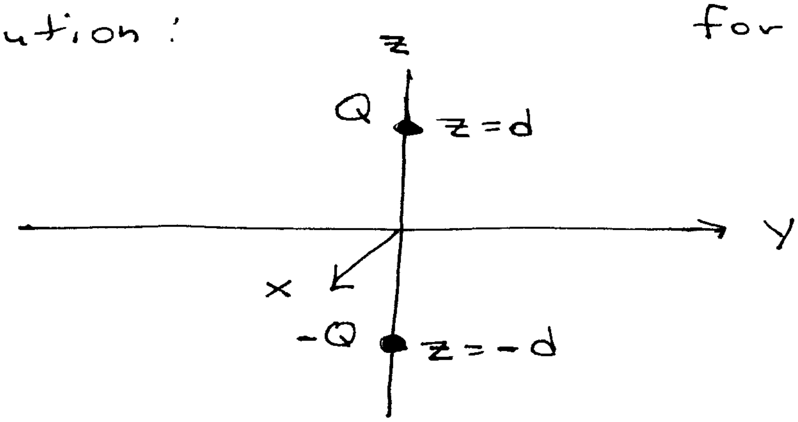
Consider a conducting plane and one charge Q:



Problem:
$$\nabla^2 V = -\frac{\rho}{\epsilon_0} = -\frac{Q}{\epsilon_0}\,\delta^3(\vec{r}-\vec{r}_d)$$

$$\vec{r}_d = d\,\hat{e}_z$$
$$V(x,y,0) = 0$$
for $z > 0$

Solution:



$$V(x,y,z) = V_Q + V_{image\,(-Q)}$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2+y^2+(z-d)^2}}$$

$$+ \frac{-Q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}$$

What is $\vec{E}$? Ans: $-\vec{\nabla} V$

What is σ on surface?

Ans: $\sigma = \epsilon_0 \vec{E} \cdot \hat{n}$ outward normal of conductor.

$$= \epsilon_0 E_z = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0}$$

What is total charge induced on surface:

$$Q_{induced} = \int \sigma(x,y)\, dx\, dy$$

$$= 2\pi \int \sigma(r)\, r\, dr$$

$$= -Q$$ (Known by Gauss's law)

What is force on Q?

Ans: $\vec{F} = Q\vec{E}$ excluding $\vec{E}$ created by Q.

What is potential energy?

$$W = \frac{1}{2}\int \rho V \, d^3x$$

with point charges, exclude self-potentials.

$$= \frac{1}{2}\int Q\,\delta^3(\vec{r}-\vec{r}_d)$$

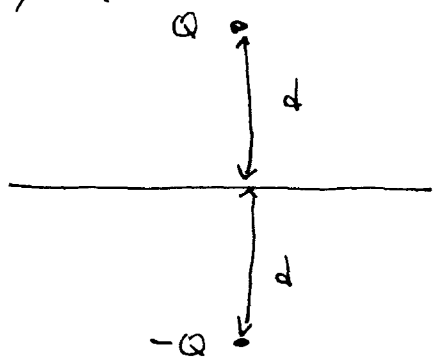
$$\times \left[\frac{-Q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2+y^2+(z+d)^2}} \right] d^3x$$

$$= -\frac{Q^2}{8\pi\epsilon_0}\frac{1}{2d}$$

(surface charge σ at $V=0$.)

Half of answer for $Q, -Q$ separated by $2d$.

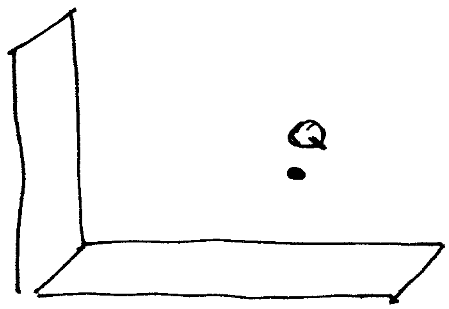
Why half?

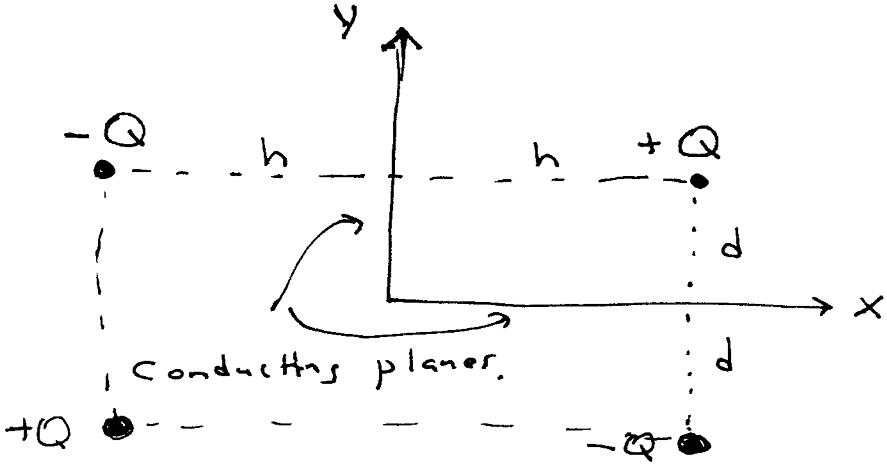


Can imagine bringing in both charges from ∞, symmetrically, both at distance d from xy plane at same time.

Conductor problem: only 1 charge needs to be moved.

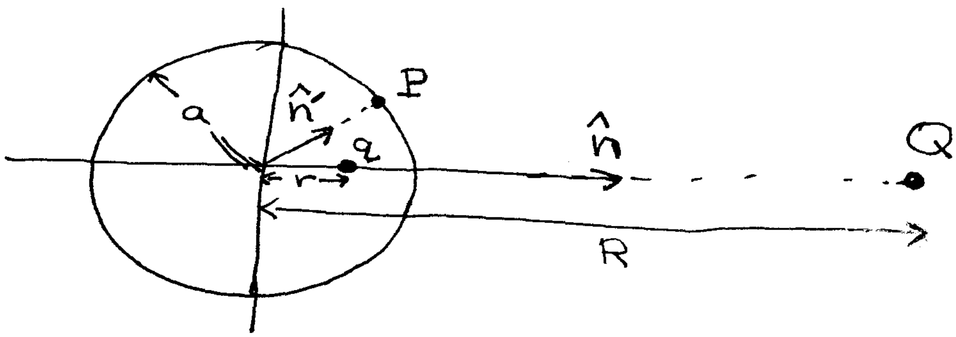
Two planes at right angles:





Works!

Grounded conducting sphere:



$$V(P) = \frac{q}{|a\hat{n}' - r\hat{n}|} + \frac{Q}{|a\hat{n}' - R\hat{n}|}$$

$$\stackrel{?}{=} \text{const.}$$

Try const = 0.

$$|a\hat{n}' - r\hat{n}| = -\frac{q}{Q}\,|a\hat{n}' - R\hat{n}|$$

Square:

$$a^2 + r^2 - 2ar\,\hat{n}\cdot\hat{n}' = \left(\frac{q}{Q}\right)^2 \left[a^2 + R^2 - 2aR\,\hat{n}\cdot\hat{n}'\right]$$

Must hold for all $\hat{n}\cdot\hat{n}'$.

$$a^2 + r^2 = \left(\frac{q}{Q}\right)^2 \left[a^2 + R^2\right]$$

$$r = \left(\frac{q}{Q}\right)^2 R$$

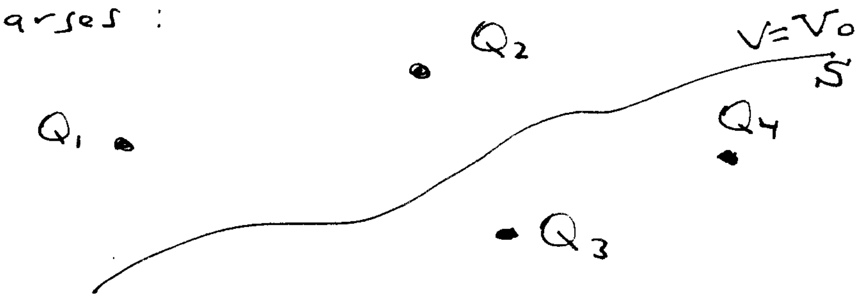
Unknowns: q, r.

Solution:

$$\boxed{\begin{aligned} q &= -\frac{a}{R}\,Q \\ r &= \frac{a^2}{R} \end{aligned}}$$

Are there other ~~solutions~~ problems with image solution?

Yes: Consider any configuration of charges:



Compute an equipotential surface $V=V_0$. Then problem with Q_1 and Q_2 + conducting surface at S with $V=V_0$ is solved by images Q_3 and Q_4.

Uniqueness Theorem

is

our lisense to guess

1. Given or assume a charge q at some point (use symmetry)
2. Guess to try a induced charge q' at $\vec{r}\,'$ 〃
3. Compute the resultand field

   a. Set the total potential V = constant at ith conductor

or b. At atleast two convenient points

or c. Set the total $\vec{E}$ ⊥ to the points on conductors

This is Method of Images

Separation of Variables

How to solve $\nabla^2 V = 0$?

In Cartesian coordinates,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Try a solution of the form

$$V(x,y,z) = X(x)\,Y(y)\,Z(z)$$

(Even if this is not general enough, sums of solutions of this form might work.)

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0$$

Now divide by $V = XYZ$

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{C_1} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{C_2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{C_3} = 0$$

Since 1st term depends only on x, 2nd depends only on y, 3rd on z, each term must be a constant.

$$C_1 + C_2 + C_3 = 0$$

$$\frac{d^2X}{dx^2} = C_1 X$$

3 possibilities:

(1) $C_1 \equiv \alpha^2 > 0$

Then $X \sim e^{\pm\alpha x}$, or $\begin{matrix}\sinh(\alpha x)\\ \cosh(\alpha x)\end{matrix}$

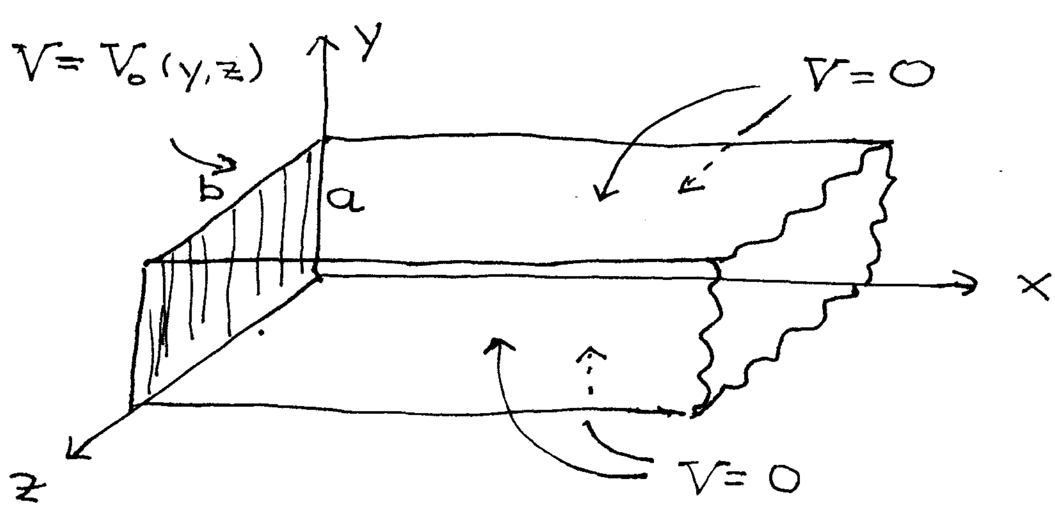
(2) $C_1 \equiv -\alpha^2 < 0$

Then $X \sim e^{\pm i\alpha x}$, or $\begin{matrix}\sin(\alpha x)\\ \cos(\alpha x)\end{matrix}$

(3) $C_1 = 0$

The $X = A + Bx$

Example (3.5 in Griffiths, pp 134-136)



Two Questions

① How do we know Sep. of Varibles will work?

② Which coordinates we should use?

(1-a) Homogeneous Linear 2nd order differential Eqs:

$$A V_{xx} + B V_{xy} + C V_{yy} \left(+ D V_x + E V_y + \overset{=0}{F} \right) = 0$$

$\Delta = B^2 - 4AC \quad > 0$ Hyperbola ; e.g. Wave Eqs.

$= B^2 - 4AC \quad = 0$ Parabola ; e.g. Heat transport

< 0 Ellipics ; e.g. $\nabla^2 V = 0$

EM, Elastic Mechanics

Potential Th, Thermo Mech, Fluid,

Each class has its own characteristic methods, features usually do not mix.

Only exception is sonic or Cherenkov Radiation:

$$\psi_{\theta\theta} + \frac{v^2}{1 - v^2/c^2} \psi_{vv} + v \psi_v = 0$$

$v < c$ or	$\Delta > 1$	Sub-sonic
$=$	$= 1$	Sonic boom
$> c$	< 0	Super sonic

(1-6) Boundary Conditions:

(i) $V(\vec{x}) = h(\vec{x})$, a given fuction at a boundary

(ii) $\nabla V \cdot \hat{n} = k(\vec{x})$, "

(iii) Mixed of the above.

Sept. of Variables works if the Eq. is

① Linear, containing no mixed term, e.g. $\sin(xy)$

② B.C. is separable in a coordinate system!

This decides which coordinates one must use!!

8.07 Electromagnetism II
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