PROBLEM SET 9 REVISED*

DUE DATE: Monday, December 3, 2012. Either hand it in at the lecture, or by 6:00 pm in the 8.07 homework boxes. (Note that the due date is being delayed because of the Makeup Quiz 2 that will take place on Thursday, November 29. Remember, however, that Problem Set 10 will be due the following Friday, December 7.)

READING ASSIGNMENT: Chapter 7 of Griffiths: *Electrodynamics*, and also Sec. 8.1, *Conservation Laws: Charge and Energy.*

CREDIT: This problem set has 105 points of credit, plus the option of earning 30 points extra credit.

PROBLEM 1: MOVING COIL IN THE FIELD OF A WIRE *(15 points)*

Griffiths Problem 7.8 (p. 300).

PROBLEM 2: MUTUAL INDUCTANCE BETWEEN TWO CURRENT LOOPS *(15 points)*

Griffiths Problem 7.20 (p. 315).

PROBLEM 3: PERFECT CONDUCTORS AND SUPERCONDUCTORS *(15 points plus 10 points extra credit)*

Griffiths Problem 7.42 (p. 334). Parts (a)-(c) are required, for 15 points. Part (d) can optionally be done for 10 points of extra credit.

PROBLEM 4: THE EFFECT OF A WEAK MAGNETIC FIELD ON THE RADIUS OF THE ORBIT OF AN ATOMIC ELECTRON *(10 points)*

Griffiths Problem 7.49 (p. 337).

PROBLEM 5: CAPACITANCE AND INDUCTANCE FOR A SIMPLE TRANSMISSION LINE *(15 points)*

Griffiths Problem 7.58 (p. 340).

PROBLEM 6: ALFVEN'S THEOREM: FROZEN FLUX IN A PERFECTLY CONDUCTING FLUID *(15 points)*

Griffiths Problem 7.59 (p. 341).

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* A typo was corrected in Eq. (8.4), which previously was missing a factor of dx on the left-hand side.

PROBLEM 7: THE POYNTING VECTOR AND THE FLOW OF ENERGY IN A TRANSMISSION LINE *(20 points)*

Griffiths Problem 8.1 (p. 349).

PROBLEM 8: MORE FUN WITH δ-FUNCTIONS *(20 points extra credit)*

In this problem you will carry out a variety of mathematical exercises intended to illustrate some of the do's and don'ts about δ-functions. I recommend that you consult the "Delta Functions" section of the Quiz 2 Formula Sheet to remind yourself of the key definitions.

(a) The Heaviside step function is defined by

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \ , \end{cases} \tag{8.1}$$

where $\theta(0)$ is often defined to be $\frac{1}{2}$. If we think of $\theta(x)$ as a distribution, an object that is defined only by the result of multiplying by a test function $\varphi(x)$ and then integrating, the value of $\theta(0)$ is irrelevant. Treating $\theta(x)$ as a distribution, prove the identity

$$\boxed{\frac{\mathrm{d}\theta(x)}{\mathrm{d}x} = \delta(x) \ .} \tag{8.2}$$

That is, assume that $\varphi(x)$ is an arbitrary smooth test function which falls off rapidly as $|x| \to \infty$. (The mathematicians often restrict $\varphi(x)$ to belong to the space of Schwartz functions, which are infinitely differentiable, and which have the property that the function and all its derivatives fall off faster than any power at large $|x|$. Here we do not expect any rigorous mathematical analysis, but instead we expect you to simply trust the standard manipulations of calculus.) Furthermore, recall that if $\Psi(x)$ is a distribution defined in terms of the integral

$$\int_{-\infty}^{\infty} \varphi(x)\Psi(x) \, \mathrm{d}x \ ,$$

then its derivative is defined by

$$\int_{-\infty}^{\infty} \varphi(x)\frac{\mathrm{d}\Psi(x)}{\mathrm{d}x} \, \mathrm{d}x \equiv -\int_{-\infty}^{\infty} \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\Psi(x) \, \mathrm{d}x \ . \tag{8.3}$$

With these definitions, show that

$$\int_{-\infty}^{\infty} \varphi(x)\frac{\mathrm{d}\theta(x)}{\mathrm{d}x} \, \mathrm{d}x = \int_{-\infty}^{\infty} \varphi(x)\delta(x) \, \mathrm{d}x \ . \tag{8.4}$$

(*Hint:* Once you figure out what you are really being asked to prove, it takes only one or two lines.)

(b) Show that

$$\boxed{\theta^n(x) = \theta(x) ,}$$ (8.5)

in the sense of distributions, for any positive integer n. (That is, show that if each side of the above equation is multiplied by an arbitrary smooth test function $\varphi(x)$, and then integrated from $-\infty$ to ∞, the two results are always equal.)

(c) The derivative of a distribution is *always* well-defined, by Eq. (8.3). Thus we might expect that we could differentiate Eq. (8.5), obtaining

$$n\theta^{n-1}(x)\delta(x) \stackrel{?}{=} \delta(x) .$$ (8.6)

Show that when combined with Eq. (8.5), Eq. (8.6) can lead to a contradiction. The bottom line is that while the derivative of a distribution is always well-defined, the product of two distributions, such $\theta(x)\delta(x)$ or $\delta^2(x)$, is generally not well-defined. ($\delta^3(\vec{r}) \equiv \delta(x)\delta(y)\delta(z)$ is however okay, because the arguments of the delta functions are independent coordinates — $\delta^3(\vec{r})$ can be thought of as a single δ-function defined as a distribution for functions on three-dimensional space.) What is

$$\frac{\mathrm{d}}{\mathrm{d}x}\theta^n(x) ?$$

(d) Show that for any smooth function $f(x)$,

$$\boxed{f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0) ,}$$ (8.7)

in the sense of distributions.

(e) If we differentiate the above identity, we find

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}\delta(x - x_0) + f(x)\frac{\mathrm{d}\delta(x - x_0)}{\mathrm{d}x} = f(x_0)\frac{\mathrm{d}\delta(x - x_0)}{\mathrm{d}x} .$$ (8.8)

The two sides do not look like they are equal, but show that they are.

(f) Evaluate

$$G[\varphi(x)] \equiv \int_{-\infty}^{\infty} \varphi(x)\delta(x^2 - a^2) \, \mathrm{d}x ,$$ (8.9)

where a is a positive constant.

8.07 Electromagnetism II

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