PROFESSOR: So here is a problem example based on Griffith's problem 9.3. So it's a two-state system, and we'll call the states a and b, with energy's Ea and Eb. And we'll call omega 0 the difference Ea minus Eb over h bar.

So those are energy eigenstates, which means that H0 is really Ea Eb. We represent the first column vector by the state a. The second state is the second column vector with entry 0 and 1-- is the state b. That's why the matrix looks like that.

And then you put a perturbation. And the perturbation, delta H, is going to take the form U delta t, where U is a matrix that has Uaa, Uab, Uba, Ubb-- the matrix elements. And they're simple enough. It's a 0 here and an alpha here, and an alpha dagger here. It should be a Hermitian matrix, and that's the way it is. The letter U, I guess, is not great, because it suggests unitary. But maybe I should put a v. I'll leave it, to not create problems with my notation.

So here is the question. This is your perturbation Hamiltonian. Actually, I wrote it wrong here. It's a delta function of t. That's delta H. So in terms of H0 applies everywhere, H0 applies everywhere, and in the middle there is a delta H for one little instant.

So what happens here? We're going to try to figure out what's going on. But the question will be posed as follows. Assume the system is in the state a for t less than 0. Let's ask the probability to be in the state b for t greater than 0. Find the probability to be, and be greater than 0.

So what's going on? Here are the states a and b. They were eigenstates of H0. If you had the system in the state a-- that's an energy eigenstate-- it would stay in the state a forever. It wouldn't do anything. Indeed if whoever prepared that state, ever since he or she prepared it, it has stayed in the state a until time equals 0.

But then the system is kicked and it's kicked with some delta function strength, and in the off-diagonal terms. That's pretty important. It's not kicked on the diagonal terms, it's kicked here. So these terms suggest a transition from state a to b here. And presumably, due to this, the system will transition.

And when it transitions, it must transition instantaneously in a sense. Because if it takes time to transition, the delta function asks for zero time. If it takes time to transition, then it would not transition. Because after the delta function has come and gone, the system doesn't change anymore. So it really has to transition instantaneously.

So at time equals 0, you must generate a probability or an amplitude to be in the state b already. So there must be an immediate transition in which the state, originally, at time less than 0 was the state a, but at time a little bit after zero must be a little bit of a plus a little bit of b. So this is t0 minus, 0 minus, and 0 plus. At 0 minus, it's a, and at 0 plus must be something like that. Because if at 0 plus it's still a, it's going to stay a forever. So it must have some amplitude to be at b at 0 plus.

So you're having your first transition amplitude problem-- how does an interaction make you jump from one state to another? Now my letters here, alpha and beta, are pretty bad-- U and V maybe-- because I have an alpha there in the interaction. And maybe, if you think perturbatively, you would say, look, if alpha is very small, there must be a small transition probability. If the perturbation is of order alpha, maybe I expect the perturbed state to be of order alpha. So maybe, after all, this is like alpha times b, or proportional to alpha times b, a number here. And this one-- well, if alpha is very small, that's probably roughly about 1 still-- a little bit less, alpha squared less, for probability conservation. But I expect an alpha b here.

So OK, when you're given a problem like that, under normal conditions it's a good idea to think about it before trying to solve it. And I think we've done a little bit of thinking. We've realized the state stays in a after a while, and immediately must transition into some other state. And once it's in that other state, its going to remain in that state forever because those are energy eigenstates, so you're going to get exponentials of E to the minus iEt. But this is going to be that.

OK, so let's solve this. Here is our equation. Let's try to write it for this case. So I would have i h bar C a dot of t. We have two states-- 1 and 2-- but here is a and b. So you would have i omega a then something, t and delta H a with something. Well, the delta H is clear that it can only be a delta H from a to b. Because there's Uab. There's only that transition amplitude. And then it would be an i omega ab, which was defined as omega 0, delta Hab of time, cb of time.

And that's the only term that exists here from the sum. You could sum over a and b, but we saw that over a, you get 0. So there's just this term. And then we have i H bar Cb dot of [? t ?] is equal to E to the i. And now you're going to have ba-- so delta Hba of t Ca of t. And here I would have omega ba, which is the opposite sign as omega ab, which we anyway call omega 0. So here's minus i omega 0 t.

Yes.

AUDIENCE: [INAUDIBLE] perturbation disappears?

PROFESSOR: Sorry.

AUDIENCE: Does the perturbation vanish?

PROFESSOR: Right. Because delta Haa is equal to 0, and therefore there's no coupling to Ca. And delta Hbb is 0 from that matrix element. And therefore, you cannot couple b to b.

And now I'll just write it i H bar Ca dot of t, i H bar Cb dot of t is equal E to the i omega 0t. Delta Hab was alpha delta of time-- Cb of t. And this is E to the minus i omega 0t, alpha star delta of time-- Ca of t. Hba was Uba, which is alpha star times the delta function of time.

Here I have a delta function of time. It says that this only matters when time is equal to 0. So things are a little delicate here. But let's see. Can I write here-- well, if I have f of x times delta of x, we know this is f of 0 times delta of x. Because anyway, only zeros [INAUDIBLE]. So I can put time equals 0 here, and forget about this exponential-- have alpha delta of t, cb of t, and here have alpha star delta of t Ca of t.

Everybody happy so far?

But here, I have Cb of t and Ca of t, and a delta t. So I should put alpha delta of t Cb at 0, and alpha star delta of t Ca at 0. Right? Yes?

People look less convinced, which I congratulate you for doubting this. This would be bad. It seems like I'm just following the logic, but I've gone too far now. Why did I go too far? First, this is going to be a disaster. Because in many ways, we don't know what these numbers are.

You see, we are argued that this is going to change this continuously. Those numbers are going to change this continuously. At time equals 0-- before time equals 0, Ca was 1 and Cb was 0. But after time equals 0, Cb is going to be a number-- something that we don't know. So the delta function, which does it hiy-- the one before 0, the one after 0, the average? What does it do? This is a case where you have a delta function that we don't know what it's doing.

So this is totally wrong. So I will just stop there. And I will consider, however these two equations.

So we have the equations now in a nice way. I'll reconsider them. And we are facing a difficulty, the difficulty of the delta function not allowing us to treat nicely the transition of i h bar Ca dot is equal to alpha delta of Cb, and i h bar Cb dot equal alpha star delta of t Ca.

Well now, the only way to resolve this difficulty is to think about this physically, and introduce a regulator that is physical. You see, no signal in the world really is a delta function. Nothing becomes infinite, and nothing lasts for zero time. So this interaction, this delta function, represents a function of time-- delta of time. It's a spike, but we can think of it as a regulated thing. You've regulated delta functions in all kinds of ways. It might be useful to regulate it as follows, as a function of time that is 0 before time equals 0, it's 0 after some time, t star, and it has height 1 over t star.

That's a picture of a delta function. It has the right area at least. And we're going to think of this system as exactly doing that in between, for t, in the interval 0 to t star. The delta function is going to be replaced by this function, this constant function. So it will have i h bar h Ca dot is equal to alpha over t star Cb. And i h bar Cb dot is equal to alpha star over t sub star. Now the star doesn't mean complex conjugation for t. It's just a number, a glyph called t0 or something like that.

And this is true for this time interval. And then you would say, look, I have two problems with this. First, you've entered this at t star, and now your calculation is going to depend on t star. And then the answer is going to depend on t star. And what are you going to put for t star? Well, that better not happen. We have to go on a limb. Many times when you do a calculation of something, you go on a limb. You say something, and see if it works. This seems reasonable. And what we expect is that we will find an answer that this independent on this t star. So that we can take this star to 0 and make sense of it.

The other question that could come is that, oh, that was your delta function. So one seconds, you used delta function for this. Maybe we have to put back these terms, these exponentials. We set them to 1. But then you could say, look, I don't think I have to put them back. Why? Because I can choose t star as small as I want, such that omega t star, which is the possible value that this can get, at most, is like 10 to the minus 60. And therefore, this phase is going to be 0, essentially, and that's going to be 1. So I claim that I don't have to worry about this anymore.

So what do we have? We have two simple differential equations, like this. With the conditions that Ca at time equals 0 is 1, and Cb at time equals 0 is 0. The state before the delta function hits is given by that. And now what we want to know is, how much is Ca after the delta function and Cb after the delta function?

So for that, we have all the tools needed. I can take this equation. This is a coupled system of equations. It's a simple system. I could differentiate again here. And I get a Ca double-dot, Cb dot, and use this equation. So you get the equation-- if you differentiate again, take a dot to form Ca double-dot, you get that Ca double-dot is equal to minus alpha over h bar t star squared Ca.

You can see that you put the i h bar here, take the derivative, you get an alpha times an alpha star, which is length of alpha squared. And the h bar appears two times, t star appears two times. This is an oscillating solution. So that's simple enough.

So what happens here is that Ca will oscillate in time, and you will have a solution of the form Ca is equal to a constant, beta 0 cosine of alpha h bar t star t, plus beta 1 sine of alpha p, over h bar t star. And if you wish, Cb from the first equation is proportional to Ca dot. And Ca dot is going to be of the form beta 0 sine off this thing, plus beta 1 cosine of this thing.

OK, so we've turned this into a tractable problem because of flattening the delta function. And your initial conditions-- again, Cb must vanish at time equals 0. So if Cb must vanish at time equals 0, beta 1 must be 0. If beta 1 is zero, this term is gone, and Ca was 1 for time equals 0. So beta 0 is 1. Therefore, Ca is equal to cosine alpha t, over h star t star Ca of t. And Cb of t, you can calculate it from the top equation. You take the derivative of Ca, and divide by alpha multiplied by t star. I'll let you do it. You get minus i alpha, over alpha, sine, alpha t over ht star.

OK, we solved for the functions. We know what's going to happen. And now, what did we want? We wanted to know what are those things for later times, for times t star or more. Now you cannot use this equation beyond time t star, because they assumed the delta function is there. So what you have to figure out is what are these coefficients at time t star. And those would be the coefficients at any later time, because there's no more Hamiltonian. So Ca at t greater than t star is, in fact, equal to Ca at t star. And Ca at t star, happily, is a number that doesn't depend on t star-- cosine alpha over h bar.

And Cb at t greater than t star is Cb at t star. It doesn't change any more after the delta function has turned off.

And Cb at t star is minus i alpha over alpha-- alpha over alpha, like this, times sine of alpha over h bar. And this is really what you wanted. Now you know what the state is doing after the delta function has turned on. It has this amplitude to be in the b state. And it is, as we predicted-- remember, this is like the face of alpha, because the alpha is alpha times its face-- length of alpha times it face, and it will cancel. And here you have sine of this quantity. So it is proportional. Cb is proportional to alpha. And that's exactly what you would expect.

So just to complete the story, let's write the answer. And what is the answer? Phi of t for t greater than 0 is the coefficient Ca of t, which was cosine, alpha over h bar, times E to the minus iEat, over h bar a. Remember, the solution is Ca E to the minus iEta. And then the other one, which is minus i alpha, over alpha, sine of alpha over h bar, e to the minus iEbt over h bar, b. That is the state after the delta function has turned on.

And what is the probability to be found in the state b, time t? It would be b times the state squared. So you get-- b will couple just to that. b with b gives you 1. This is a phase. This is a phase. Sine squared of this thing is sine squared of alpha over h bar. What is the probability to be in a? It's a psi of t. And in this case, it will be cosine squared of alpha over h bar.

And it's very handy that cosine squared plus sine squared is equal to 1, because it has to be either in a or on b. And it's kind of interesting, if you were doing this in perturbation-- we solved it exactly. If you were doing this in perturbation theory, we would have found P of b proportional to alpha squared. Because the first term in the expansion here, in terms of the interaction.