PROFESSOR: So again, much of the difficulty here is just making sure you're doing the perturbation theory right and working with the two basis correctly. But now we're in the good position. We can combine our results. So here is what I want to combine.

I want, in principle, to combine all the things, all the terms. And we've calculated all the terms to some degree. So we saw the Darwin term only affects l equals 0.

On the other hand, spin orbit really requires s dot l. And the state should be acted nontrivially by l. If all your states are singlets, l gives 0 on them. So actually, you have a little bit of an uncomfortable situation because l acting on l equals 0 states will give you 0.

But you still have the 1 over r cubed. And the 1 over r cubed has l dependencies here. So it's 0 in the denominator, 0 in the numerator. The whole spin orbit coupling doesn't seem to make sense for l equals 0. So most people say, physically, the spin orbit coupling vanishes for l equals 0. It should not be there.

But then something very funny happens. If you take a proper limit of this as l goes to 0, it gives you, for the spin orbit result, the same thing as the Darwin result. So for l equals 0, the spin orbit limit is actually the same as the Darwin. It's almost as if you say, oh, the spin orbit has everything in it. But it's not legal for l equals 0. But the Darwin does it.

So if I put together the spin orbit and relativistics for l different from 0, I'm legal. For l equals 0, I should really sum relativistic and Darwin. But actually, it turns out that it's the same as summing relativistic plus spin orbit. Because the limit of spin orbit for l equals 0 is equal to the Darwin.

So we will just sum relativistic plus spin orbit. And that gives the result for everything, including Darwin for l equals 0. So this is minor subtlety. But the end result is that we can combine it.

So the happy thing, as well, is that we now can do this. mj delta H relativistic plus delta H spin orbit nljmj. Because whatever we calculated here was actually the same that H relativistic in the coupled basis.

And anyway, the spin orbit, we use the coupled basis to get this number. So this thing, you can add the two results, and you get En0 squared over 2mc squared 3 plus n J j plus 1 minus 3l l plus 1 minus 3/4 over l l plus 1/2 l plus 1-- a little messy.

OK. Something very unexpected happens now. It's something that when you do the algebra yourself, you say, wow. How did that happen? Here is the issue. We are trying to compute the splittings of the hydrogen atom. And for this, as we said in the coupled basis, the degeneracies happen for a given value of n.

I'm getting good at doing this diagram now. So we're looking at the fixed n. And then, you have all kind of degenerate states for various l's and j's. So is there a better way to rewrite this formula?

And you say the following-- suppose you think of some states of fixed j-- fixed j states-- j. For example, this state-- we'll use our l equals 1 over here-- I have 1, 2, 3-- 3P 3/2 and 3P 1/2. Here, I have 3D 5/2 because this is l equals 2. So 2 plus 1/2 and 2 minus 1/2.

So I have here states of fixed j-- two states with the same j. But they come from different l's. Because when you get a total j of some value, it can come from a lower l-- l plus 1/2 gives you that j-- or it can come from a top l with l minus 1/2 giving you that j.

So a given j value can arise from an l that is 1/2 higher or an l that is 1/2 lower. So for a fixed j, it may be that l is equal to j minus 1/2, or l is equal to j plus 1/2.

Now, look at this quantity. We'll call this whole quantity f of jl. Very astonishingly, f of jl when l is j minus 1/2, or f of jl where l is j plus 1/2, you can calculate it.

Put l equals j minus 1/2 on that formula. And then put l equals j plus 1/2 in that formula. You would say, it's going to be a mess. In fact, both cases are the same. And it give you minus 2 over a j plus 1/2.

So the whole l dependence here, amazingly, is fake. There's no l dependence in this factor. It is a little strange. There's no l dependence because, given j, l can be two values. And for those two values, that function gives you the same. It's one of those, like x squared gives the same for 1 and minus 1.

This function, once you fix j, l can be two values. And it so happens that these two values give the same. So at the end of the day, this just depends on j. That's the most important result of this lecture. The whole structure, once you put relativity, Darwin, and spin orbit, just depends on j.

And what is the result when you simplify this? Our result is that the fine structure Enljmj-- fine structure 1-- is equal to minus alpha to the fourth mc squared 1 over 2n to the fourth n over j plus 1/2 minus 3/4. Whole answer, all together-- Darwin fine spin orbit and relativistic.

So a few comments about this. People that look at the Dirac equation more seriously would have expected this result. It turns out that in the Dirac equation, the symmetry and the rotations-- the generator of rotations-- is exactly j, which is l plus s. That generates rotation. That commutes with the Hamiltonian. So you should expect that the energy eigenstates are j eigenstates.

So here, we're seeing that, yes, they can be simultaneously diagonalized. The eigenstates can also be labeled by the j value. So the exact eigenstates in the Dirac equation can be labeled by the j value, and we're seeing a reflection of this.

This means that the j multiplets are not going to be split, and there's going to be, moreover, some degeneracy. They're not split because there's no mj dependence. And they are all going to be the same.

So let me finish by drawing how the spectrum looks. It's pretty important to see that. This quantity over here, this numerical quantity, is always positive-- positive-- for any state in the hydrogen atom. You can see that because j max-- the maximum value-- so this is negative.

So you need to know what is the minimum value of this quantity to see if it goes negative. The minimum value of this ratio is when j is maximum. The maximum j is when j is l plus 1/2 with the maximum l.

So this is l plus 1 maximum. But l plus 1 maximum is, in fact, n. So the minimum value of this is 1. So this is always positive.

As n increases, the corrections become smaller. So what happens? All the corrections are negative. So if you had the original states, they all go down a bit. Even the ground state goes down a bit. 1S 1/2 is down. Because for n equal 1, you do get a state.

Then what do you get? 2S 1/2 is here. It's also down a bit. 3S 1/2-- so this is l equals 0. Then you have l equals 1. Remember, what did we have here? We have 2P 1/2 and 2P 3/2. Because l equal 1 gives you j equals 3/2 and 1/2.

Look, these two remain degenerate because they have the same n and the same j. And that's

all that matters. So 1/2 and 1/2 have the same j, so they remain degenerate. 2P 3/2 has a higher j, therefore, has a smaller number. So it's lowered less, and it appears a little higher.

So here, you would have 3P 1/2 and here, 3P 3/2. And the next state here is 3D. Now you're combining l equal 2. So you get 5/2 and 3/2. The 3/2 is degenerate with this 3/2 because it has the same j. And the 5/2 is a little higher-- 3D 5/2.

So this is our final picture. The hydrogen atom, all the states get pushed down. The various multiplets with different l but the same j are still degenerate. This formula has no l dependence. So these two are the same. These two are the same. These two are the same.

Moreover, the j multiplets are not split. Every j multiplet differs [? because of ?] a collection of states with different mj. But mj doesn't appear. So this is your fine structure of hydrogen atom.

And this is where you study in detail the Zeeman splitting and the Stark splitting. And we'll talk a little bit about them next time, as we will begin, also, our study of the semiclassical approximation. See you then.