PROFESSOR: So far, so good. We've really calculated. And we could stop here. But there is a nice interpretation of this Darwin term that gives you intuition as to what's really happening. See, when you look at these terms overall-- these interactions with this class-- you say, well, the kinetic energy was not relativistic. That gives rise to this term.

There is a story for the spin orbit as well. We think of the proton-- creates an electric field. You, the electron, are moving around. As you move inside the static-electric field, you see a magnetic field. The magnetic field interacts with your dipole moment of the electron. That's the story here.

So what's the story for this term? Where does it come from? What is the physics behind it? That's what I want to discuss now. So the physics interpretation of the Darwin term is that the electron behaves not as if it would be a point particle but as if it would be, kind of, a little ball where the charge is spread out.

We know that any mechanical model of the electron, where you think of it as a ball, the spinning doesn't work for the spin. But somehow, here, we can see that this extra correction to the energy is the correction that would appear if, somehow, the electron, instead of feeling the potential of the nucleus at one point, as if it would be a point particle, it's as if it would be spread.

So let me make a drawing here. Here is the proton. And it creates a potential. And we will put an electron at the point r. There's the electron. But now, we're going to think that this electron is really a cloud, like this. And the charge is distributed over there. And now, I'm going to try to estimate what happens to the potential energy if it's really behaving like that.

So for that, I'm going to make another coordinate system sort of starting here. I'm going to call the vector that goes to an arbitrary point here vector u. So from the origin or center of this charge distribution, we put the vector u. And then I have this vector over here, which is r plus u. And this u points to a little bit of charge, a little cubit here, for example.

So this is our setup. Now, the potential-- due to the proton-- proton-- we have a potential V of r, which is equal to minus the charge of the electron times the scalar potential, phi of r-- it's minus e times e over r, which is our familiar minus e squared over r. So the potential energy-- this is the potential energy.

And so if you had a point particle-- for a point particle-- you have that the proton creates a potential at a distance r. You multiply by the charge. And you get the potential energy. So how do we do it a little more generally?

I'm going to call this V tilde of r. The true potential energy-- potential energy-- when the center of the electron is at r-- electron at r. So what is the true potential energy when the center of the electron is at r? Well, I would have to do an integral over the electron of the little amount of charges times the potential at those points.

For every piece of charge, I must multiply the charge times the potential generated by the proton. And that would give me the total energy of this electron in this potential, V of r. So let me use that terminology here. I will describe the charge density-- density-- rho of u-- the density of charge at a position given by a u vector.

I'll write it as minus e times rho 0 of u-- a little bit of notation. I apologize, but it's necessary to do things cleanly. The charge density of the electron-- electron-- is given by rho of u minus e rho 0. And then it should be true that the integral over the electron of d cube u rho 0 of u is equal to 1.

Look at that integral. Why should it be equal to 1? It should be equal to 1, because then the integral of the charge density-- the integral of this over all of space would be minus e times 1. And that's exactly what you expect. So rho 0 is a unit free-- well, 1 over length cubed is a charge-free quantity. The charge is carried by this constant over here.

So what happens then with this V of r? This V of r is the charge-- little charge q is d cubed u times rho of u times phi at r plus u. Look, the charge is this at some little element. And then the potential there is the potential at r plus u.

So one more step-- V of r-- if you take from rho this extra minus e, minus e times phi of r is what we call the potential energy due to r. So take the minus e out and append it to the capital Phi here, so that we have integral d cube u rho 0 of u V of r plus u.

OK, this was our goal. It gives you the new energy-- the true potential energy of this electron when this center is at r as the smearing of the potential energy of a point particle smeared over the electron. So this is a formula that represents our intuition. And moreover, it has the rho 0 here that tells you the weight that you should apply at any point, because this rho 0 is

proportional to the charge-- very good.

We have a formula. But I was supposed to explain that term. And we have not explained it yet. But now, it's time to explain it. We're going to try to do a computation that helps us do that. The idea is that we're going to expand the potential around the point r and treat this as a small deviation because, after all, we expect this little ball that I drew big here for the purposes of illustration to be rather small.

So let's write V of r plus u as V of r in Taylor series-- plus the next term is the derivative of V with respect to position evaluated at r multiplied by the deviation that you moved. So sum over i dv dx i evaluated at r times u i. This is the component of this vector u. That's the first term in the Taylor series.

And we need one more-- plus 1/2 sum over i and j, d 2nd V vx i vx j evaluated at r ui uj. And then we have to integrate here. We substituted in here and integrate. So V tilde of r-- let's see what it is.

Well, I can put this whole thing-- we don't want to write so much. So let's try to do it a little quick. We're integrating over u. And here, let's think of the first term here. When you plug in the first term, this function of r has nothing to do with u. So it goes out and you get V of r times the integral d cube u rho 0 of u.

For the next term in here, these derivatives are evaluated at r-- have nothing to do with u. They go out. So plus sum over I dv dx i of r integral d cube u rho 0 of u, ui. Last term-- these derivatives go out. They're evaluated at r. They have nothing to do with you.

So plus 1/2 sum over i and j d second V dx i dx j evaluated at r integral d cube u rho of u ui uj. OK-- all these terms! Happily we can interpret much of what we have here. So what is this first integral? V of r equals-- this integral is our normalization integral. Over here, that's equal to 1.

So this is very nice. That's what you would expect that to first approximation, the total energy of the electron, when it is at r-- as if it would be a point particle at r. Now, let's look at the next terms. Now, we will assume that the distribution of charge is spherically symmetric. So that means that rho of u vector is actually a function rho 0 of u, where u is the length of the u vector.

So it just depends on the distance from the point that you're looking. That's the charge. Why would it be more complicated? If that is the case, if this is spherically symmetric, you can

already see that these integrals would vanish, because you're integrating the spherically symmetric quantity times a power of a coordinate. That's something you did in the homework as well for this Stark effect.

You realized that integrals of spherically symmetric functions then powers of x, y, and z-- well, they get killed, unless those are even powers. So this is an odd power. And that's 0. So this term is gone. And this integral is interesting as well.

When you have a spherically symmetric quantity and you have i and j different, the integral would be 0. It's like having a power of x and a power of y in your homework. So if that integral would be 0, it would only be non-zero if i is equal to j-- the first component, for example-- 1 u1 u1. But that integral would be the same as u2 u2 or u3 u3.

So in fact, each of these integrals is proportional to delta i j, but equal to 1/3 of the integral of d cube u rho of u-- I'll put the delta ij here-- times u squared. u squared is-- u1 squared plus u2 squared plus u3 squared-- each one gives the same integral. Therefore, the result has a 1/3 in front.

So what do we get? Plus 1/6-- and here, we get a big smile. Why? Because delta i j, with this thing, is the Laplacian of V evaluated at r. They have the same derivative summed. So what do we get here? 1/6 of the Laplacian of V evaluated at r times the integral d cube u rho 0 of u u squared.

Very nice-- already starting to look like what we wanted, a Laplacian. Can we do a little better? Yes, we can. It's possible. Let's assume the charged particle has a size. So let's assume the electron is a ball of radius the Compton wavelength, which is h bar over mc.

So that means rho 0 of u is non-zero when u is less than this lambda. And it's 0 when u is greater than this lambda. That's a ball-- some density up to there. And this must integrate to 1. So it's 1 over the volume of that ball, 4 pi lambda cubed over 3.

Do this integral. This integral gives you, actually, 3/5 of lambda squared with that function. So the end result is that V tilde of r is equal to V of r. Plus 3/5 of this thing times that is 1/10 h squared m squared c squared Laplacian of V.

The new correction to the potential, when this mole came out to this 1/10-- that's pretty close, I must say. There's an 1/8 there. We assume the electron is a ball of size, the Compton

wavelength of the electron. It's a very rough assumption. So even to see the number not coming off by a factor of 100 is quite nice.

So that's the interpretation. The known locality-- the spread out of the electron into a little ball of its Compton size is quite significant. The Compton wavelength of the electron is fundamental in quantum field theory. The Compton wavelength of an electron is the size of a photon whose energy is equal to the rest mass of the electron. So if you have a photon of this size equal of wavelength equal to Compton length of the electron, that photon packs enough energy to create electron-positron pairs.

That's why, in quantum field theory, you care about this. And the quantum field theory is really the way you do relativistic quantum mechanics. So it's not too surprising that, in relativistic analysis such as that of the Dirac equation, we find a role for the Compton wavelength of the electron and a correction associated with it.