

**PROFESSOR:** OK. Let me then say a little more about that extra term. Let's rewrite it in a slightly different way so that we get the feeling of what it has to do with Hamiltonians.

So this is a coupling term. So we try to understand, what is that coupling  $\psi_k, \dot{\psi}_n$ ? How could we write it? So what is  $\psi_k, \dot{\psi}_n$ ?

To figure this out, we have no option, probably, except looking at the Schrodinger equation-- not really the Schrodinger equation-- the time-instantaneous state conditions--  $E_n$  of  $t, \psi_n$  of  $t$ . So you look at this instantaneous state condition for  $n$  and say, OK, I'm going to differentiate, and then I'm going to make the overlap. What else could you do?

This is the equation that tells you how the instantaneous energy eigenstate is supposed to change in time. So it must have the information of how to compute this, or how to rewrite it in terms of other things that are interesting. So let's differentiate with respect to time.

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And we use dots, as well. So let's write this as  $\dot{H} \psi_n + H \dot{\psi}_n = \dot{E}_n \psi_n + E_n \dot{\psi}_n$ . And then we do what-- we want  $\psi_k$  here from the left. So we'll have a  $\psi_k, \dot{H} \psi_n +$

Now you're going to put the  $\psi_k$  here. The  $H$  will act on the left. So we will have an  $E_k \psi_k, \dot{\psi}_n$ . That's all that is left from this term. The  $\psi_k$  came from the left and was acted by the  $H$  to give you an  $E_k$  times this.

On this state, I have the  $\psi_k$  already hitting the  $\psi_n$ . Now, we are interested in this term when  $k$  is different from  $n$ . These are the couplings to other states in the analysis of the adiabatic approximation.

So let's take  $k$  different from  $n$ . In which case, this term will give 0. Because  $k$  is different from  $n$  and the state just hits it. And here we have, finally,  $E_n \psi_k, \dot{\psi}_n$ .

So good. I think we can solve for this state in terms of a matrix element of the time derivative of the Hamiltonian. So what do we have? We have that  $\psi_k, \dot{\psi}_n$  is equal to  $\frac{1}{E_n - E_k} \dot{H}_{kn}$ .

The  $kn$  matrix element of  $\dot{H}$  is not equal to the time derivative of the  $kn$  matrix element of

H. You calculate the  $\dot{H}$ . And you put the  $k_n$  if you wish. It really is this thing-- the matrix element of  $\dot{H}$  over  $E_n$  minus  $E_k$ .

The nice thing of this representation is that it now begins to give you a feeling of why we care even about slow changes. We have, in this coupling term-- this is the coupling term that will take you away from an instantaneous energy eigenstate. This is the term that can ruin your adiabatic approximation.

And that term, now, you understand that it has to do with the rate of change of the Hamiltonian, which, after all, is the thing that drives the physics. You could say that term must be small, because the instantaneous energy eigenstates must be slow. And that's true. But this equation makes it clear that you have here a matrix element or some information about the time rate of change of your Hamiltonian. And if that is slow, then this term is small. And presumably, a small term in a differential equation has little effect.

So probably, it's a good idea to just represent a particular case of a Hamiltonian to just complete this discussion. So imagine you have a Hamiltonian that-- it's of this kind, varies a bit. Maybe it doesn't vary for time equal less than 0.

It varies for a while, and then it stops varying. And let's call the time capital  $T$ -- the time for the variation. This is the typical thing that people do in the adiabatic approximation. So that is the language of things-- nothing, variation during a time  $T$ , nothing after all.

So what is the goal, the result here? What is the real statement at the end of the day is that the errors that appear in the adiabatic approximation are all proportional to  $1$  over  $t$ . So that is a mathematical statement. The  $1$  over this capital  $T$  controls all the errors. So if the change happens very slowly, the errors go to  $0$ , like  $1$  over  $t$ .

And we can see the beginning of that, because I could make a model of this Hamiltonian that is a little different. It just may be a line growing here, for example. So this is a Hamiltonian  $H$  of  $t$  that begins as  $H_0$  plus  $t$  over capital  $T$  times  $V$ . That is for  $t$  between  $0$  and capital  $T$ , and then  $H_0$  plus  $V$  for  $t$  greater than that capital  $T$ . So here is  $H_0$ . Here's  $H_0$  plus  $V$ .

So if you have this thing here, you will have that  $\dot{H}$  is equal to  $V$  over  $T$ -- capital  $T$ . And that's intuitive. The more time you take to make this finite transition, the smaller the value of the derivative. And therefore, an  $\dot{H}$  matrix element  $k_n$  will be  $1$  over capital  $T$ , the matrix element  $k_n$  of the operator  $V$ .

This term, this overlap, will be proportional to  $1/T$ . And this error term in the differential equation will be proportional to  $1/T$ . That takes still some effort to show that when you integrate the differential equation, the error remains  $1/T$ . It doesn't get increased. But that is precisely the statement of the adiabatic theorem.

The adiabatic theorem says that-- and I'll quote it, maybe, this way. If you have the norm of a wave function, you can define the square root of the overlap of the wave function. And the way people precisely state the adiabatic theorem is that. The state  $\psi$ , during a time  $t$  minus the  $\psi$  adiabatic-- the  $[\psi_n]$  for the adiabatic state of time  $t$ . The true-- the different-- I'm sorry. The difference between the true state  $\psi$  of  $t$  and the adiabatic approximation  $\psi_t$ , for  $t$  between 0 and  $T$ , is less than or equal than a number of times  $T$ .

So the adiabatic approximation that we've written, the state with all these phases, is as close to the real Schrodinger equation solution-- the error is less than  $1/T$ , where  $T$  is this time in which you have nothing here, nothing there. And this, in fact, gets even better for larger times. So after larger times, the error doesn't increase. So you can keep-- the change has already happened, and it doesn't matter.

So basically, we've argued intuitively what the adiabatic approximation should be by tracking an instantaneous eigenstate. We've now done it with explicit differential equations. If you have an explicit problem, you can, after all, solve this and see what things happen, and therefore, reach the conclusion that then there are a good number of conditions. Many situations, they're slowly varying. The adiabatic  $[\psi_n]$  will give a good solution to the Schrodinger equation.

So that completes our first part of the analysis. Now we're going to turn into a little different story.