MITOCW | L12.1 Ionization rate for hydrogen: final result

PROFESSOR: Today, let's catch up with what we were doing before. And last time, we were talking about hydrogen ionization. And we went through a whole discussion of how it would happen. It was ionized because there would be an electromagnetic wave within the hydrogen atom. The initial state was the ground state of the hydrogen atom. The final state was a plane wave.

Our main work was computing this matrix element, and that's what we did last time. It took us some work, because there is a spatial integral here that was quite complicated, given the various directions that are going on. But that was our result. Here is the formula. And a few things to notice here were that there's an angle theta that we recognized as the angle between the electric field polarization and the momentum of the electron.

So k is the momentum of the electron, and when k doesn't have a narrow, it's the magnitude of the electron momentum. As any k-- well, it's a little bit of an exaggeration. h bar k is a momentum. But we understand when we say that k's a momentum. k times a, it has no units, which is appropriate of the units of this quantity, our units of energy. And then another quantity here, E naught, is the magnitude of the electric field defined by this formula, with a 2 E naught here.

We discussed that the photon had to have an energy that it was not supposed to be too big, so that the wavelength of the photon would be smaller, or much smaller than the atom, in which case the spatial dependence of the electromagnetic field would be relevant. So for simplicity, with took photons not to be too energetic. And we also took photons to be energetic enough that, when they would ionize the atom, the electron that would go out would not be too affected by the cooling potential, and we could treat it like a plane wave to a good approximation-- a free plane wave. Otherwise, you would have to use more complicated plane waves associated with a hydrogen atom.

So under this ranges, these ranges translate to this ranges for the momentum of the electron. And if the-- as required, the energy of the photon is significantly bigger than a Rydberg, which is the 13.6 eV. This is like 10 Rydbergs in here. Then the momentum of the electron is given by this formula to a good approximation. It's essentially the energy of the photon, then the square root of that.

The last ingredient in our computation is the density of states. This was calculated also a couple of weeks ago, so you have to keep track of some formulas here. There's some formulas that they're a little complicated, but we'll have to box them and just be ready to use them, or to spend five minutes really writing them. And this was the formula for the density of states with some energy E. With that energy E, these were free plane wave states. Energy E, momentum k associated with E, and being shot into some solid angle in space. And this is the solid angle. So I wrote it as a solid angle in here.

So this is basically where we stood and, at this moment, we want to complete the discussion and get a simple expression, and simplify it, and get what we need to have for the ray. So, if you remember, Fermi's golden rule expresses a rate omega in terms of 2 pi over h bar, rho parenthesis E, times Hfi squared. So that's Fermi's golden rule. And this is what we want to apply.

Well, we've calculated the matrix element. This would be Hfi prime here. That's what we have up there. We have rho. We have this quantity, 2 pi over h bar is just a constant. So we can simply plug this here, square that. There's lots of h bars and things that I don't think you'll benefit if I go through them in front of your eyes. So I'll write the answer. But in writing the answer, remember this rho is a differential, in some sense. Some people might put you in a d rho here, because there is a d omega.

So think of this, when I substitute rho, I have the d omega, and I think of this as the d w, a little rate. So dw, d omega. So I simply pass from w to I called it dw, and substituted the rho here for d omega. And here is what we get-- 256 over pi, e E naught a naught squared over h bar, m a naught squared over h bar squared, k a naught cubed over 1 plus k a naught squared to the 6th, and cosine squared theta.

OK, it's still a little complicated. But it's mainly complicated because of constants that have been grouped in the best possible way, in my opinion, to make it understandable. This is an energy. And this is an h bar. This is actually a Rydberg with a factor of 2. So in a second, you can see that this thing has units of 1 over time.

There's also an important factor here, and that's an intuitively interesting fact, that the emission of the electron is preferentially in the direction that the electric field is polarized. There's a cosine squared theta. There's no electrons emitted orthogonally to the electric field. And that's kind of intuitive. It's almost like the electric field is shaking, the electron, well, it kicks it out.

OK, this is a differential rate. So the total rate w is the integral of this differential rate over solid angle. And for that, you need to know that the integral over solid angle of cosine squared theta is 4 pi. If you didn't have the cosine squared, you would have the 4 pi. But usually cosine squared, this is multiplied by 1/3. You can do the calculation. It just doesn't take any real time.

But many people remember this by thinking of the sphere, or planet Earth. Cosine squared theta is large near the North Pole. It's large near the South Pole. That doesn't amount to much. As opposed to sine squared theta, which is large all over the big equatorial region. So it's a little bigger, and it turns out cosine squared, the average over the sphere is 1/3. And sine squared, the average over the sphere is 2/3. Well, it's kind of not a bad thing to know, because it saves you a minute or two from doing this integral.

And now, I'm also going to apply something that basically-- you know, a formula sometimes gives you more things than you should really trust. And I would say here, this 1 is not to be trusted basically, because k a naught must be significantly bigger than this number probably for this to be accurate. So under most circumstances, this 1 is not worth it. In fact, if you calculate this thing using different approximations, people sometimes don't get this 1. And you may see that in books.

So let's ignore this 1. And then this answer is 512 over 3 e E naught a naught squared, over h bar Rydberg, 1 over k a naught to the 9th. Pretty high power. And the answer starts to be reasonably simple. This 9 arises because of 12 minus 3 here.

This is still probably not ideal, if you want to play intuition about what's going on, and the scale of the effects. You know, once you have a formula, and you've worked so hard to get it-- this calculation is doing it reliably is a couple of hours of work-- and you might as well manipulate it and try to make it look reasonably nice. And this is what people that do atomic physics do with this rate.

So they write it in the following way. Again, a short calculation to get this 256 over 3. And they put in atomic units, Ep over E star-- I will explain what these numbers are-- squared, 1 over t star, 1 over k a naught to the 9th.

I think it's important to notice as well that you could say, OK, so how did the photon energy, or photon frequency, affect this result? Well, actually, the photon omega doesn't seem to be anywhere here. It has disappeared. But it is implicit in the electron momentum, because basically the momentum of the electron is what is obtained from using the energy of the photon. It ionizes the electron, liberates electron, and then gives it some kinetic energy. And, roughly, k equals like the square root of omega. So there is an omega dependence here.

Now, what are these other quantities? These quantities are simple. Ep is the peak electric field. It's peak electric in the wave that you've sent in. So it's 2 E naught. E star is the atomic electric field. And that's defined as the electric field that the proton creates at the electron. So it's equal to E over a 0 squared. That's the definition.

Or, if you want, in terms of Rydbergs, 2 Rydbergs over E a naught, and it's about 5.14 times 10 to the 11 volts per meter. So it's a nice quantity. It's in the land you're comparing your electric field from your laser to the typical electric field in the atom. And that ratio is meaningful. And then t star is the time the electron takes to travel a distance a naught. So it's a naught over the velocity of the electron, which is roughly the fine structure constant times c.

You remember the electron roughly has a beta parameter equal to 1 over 137. This is [INAUDIBLE] alpha equal beta, meaning the fine structure constant in alpha is the beta of the electron. So this is t star. And it's actually two Rydbergs over h bar, and it's 2.42 times 10 to the minus 17 seconds. That's the time the electron, which is not moving that fast, takes to move a Bohr radius.

So this is the last form we'll take. Atomic physics books would consider that the best way to describe the physics of the problem. And this is really all we're going to say about ionization. It's kind of a precursor of field theory calculations you will do soon-- not in 806-- in which you do reasonably complicated calculations, matrix elements, Feynman diagrams. And, at the end of the day, by the time you're all done, the answer simplifies to something rather reasonable, and not complicated.