PROFESSOR: So I'm going to do today this identical particles. We'll continue with identical particles. We'll get almost to the resolution of the issue of exchange degeneracy.

In order to do this identical particles properly, you have to do a bit of group theory or understanding what permutation operators are well. So it allows you to think about these matters more clearly. So that's what we have to do today.

So let me review. So we were doing identical particles. And we spoke of exchange degeneracy. We had, say, for the state of two spins-- identical spins-- we said could be represented by this state or by this state in which you exchange the pluses and minuses. So the problem of exchange degeneracy is that, for identical particles, you would almost say that these two states are equivalent.

One particle is in plus 1 particle, is in minus. They're identical. So you cannot tell whether it's the first or the second. So you would say these two states are equivalent.

But we saw that that leads to a contradiction. If these two states are equivalent, any linear superposition of them should be equivalent. And then when we ask the question of the probability of finding these states in a particular state that was along the x direction-- both spins along the x direction-- we found that it depended on what linear combination you took here.

So it's just not consistent with the principles of quantum mechanics to say that these two states are equivalent. So they're not. And we better understand how we can solve this problem.

Now, I think the solution, in a sense, you've heard before. And you know how you're supposed to work with identical particles. But I think going through this in detail and seeing what is the structure of operators and Hilbert spaces that leads to that solution or why that solution is the unique solution to this problem, it's interesting. So we begin with permutation operators. Operators.

So we consider a two-particle system. We'll begin with two particles. And then we'll eventually generalize to n particles. We'll do everything for two particles first.

So the particles, they will live on a vector space, V, each one so that the set of two particles lives on V tensor V. So each particle has a set of possible states that are the vectors in V. And the two particles have a set of possible states that are the vectors in V tensor V.

Now, for the moment, whether they're identical or not identical will play no role. It will play no role throughout this lecture, where we understand those operators in Hilbert spaces. So it will not matter.

So the particles may be distinguishable, may not be distinguishable. And we will write states here in V tensor V, based as states, for example, as ui1 tensor u2-- I'm sorry, uj2. So with little i and little j, they're noting different integers. Those are different basis vectors.

And this is a set of basis vectors in V tensor V-- so basis vectors for all i and j. Now, sometimes we get a little lazy, and we stop writing the 1 and 2. And we write it as ui tensor uj. Or even we can get more lazy and just write ui uj. So if we do that, you should know you really mean this whole thing.

So is a permutation operator? It's a linear operator that does some funny thing on the state. So let's look into it. So a permutation operator, P2,1 That's a name I will give it. P2,1 is a linear operator acting on the space V tensor V-- so linear operator.

And this is called the permutation operator. So to know what this permutation operator is, you need to know how it acts on a basis state. So let's put one of these basis states, ui1 tensor uj2.

And this permutation operator, you just need to know how it acts in the basis state. That's the safest way always to know that you have a linear operator. If you know how it acts on a basis state, then on any sum of basis states, you know, because it's linear.

On the other hand, if somebody tells you, this operator does this to all vectors-- something-- maybe it's not a linear operator. It may be a more complicated thing. So here, what it will do, it will put the state that was in 2 in position 1, and put the state that was in 1 in position 2. So it will take you j1 tensor ui2.

So the state in 2-- this first thing that says that the state in 2-- should be put in position 1. So the j label-- it was uj. Now it's in position 1.

And the state that was in 1 should go to position 2. The one thing we never do is flip these things. The first particle goes before the second particle. That's our order of things. But the state has moved.

OK. So this is the prescription of what this does. And then you realize already one property-- that P2,1, if you let it act twice in a state, it will flip the ij the first time. And it will flip them again the second time.

So this is the unit operator. So it's inverse. So that's a nice thing.

Now, perhaps a little less obvious, but kind of still simple, is that P2,1 is Hermition. How do you check that an operator is Hermition? An operator, M, is hermission if you have, for example, M alpha beta is equal to alpha and beta. The M operator moves from this position to the other with an M dagger. But if M is Hermition, it will work like that.

So let's try this with the operator P2,1. So we'll try a P2,1 on the state alpha, and then [INAUDIBLE] with a state beta. So we need a number of letters.

So let's do it here. P2,1 on a state alpha, which will be ui1 uj2, [INAUDIBLE] with a state beta uk1 ul2. OK. Here it is. P2,1 acting on some state, alpha, [INAUDIBLE] with a state beta. We'll evaluate it.

Well, we can never write it quickly, I think. This flips the j and i so the j is now on the first particle and the case in the first particle. So the inner product, assuming this is an orthonormal basis, is just delta jk.

And here, this operator moves the i label to position 2. And here you have an l in position 2. Therefore, it's a delta il.

On the other hand, we can calculate this ui1 uj2, and now put the P2,1 operator here between the uk1 and the ul2. This time you flip the k and l, so the l ends up being the first Hilbert space, where there's an i. So you get delta li.

And the key, ends up in the second Hilbert space, where you have a j here. So you have a j with a k. And these two are the same.

And intuition-- of course, I've written all these equations. But intuition is, if you have an inner product-- this chronicle delta sign-- you exchange two here. The result is the same as if you exchange two here. It will do the same thing.

So it's a Hermitian operator. So operator P2,1 is Hermitian. That means that the relation P2,1 P2,1 equal 1. Since this is Hermitian-- this P2,1 dagger-- P2,1 equal 1.

So we learn that also the operator is unitary. P2,1 is unitary. That's nice, because that means that could be a symmetry in quantum mechanics.

Operators in quantum mechanics that preserve the norm of states can be symmetries. And P2,1 is unitary. OK. So we have our operator P2,1. Let's do things with it.

So we can define two things with P2,1. Define this operator, S, which will be 1/2 1 plus P2,1, and the operator, A, which is 1/2 1 minus P2,1. And here comes the claim. These two operators are orthogonal projectors.

They're orthogonal projectors. Projectors. So let's review what that means. So basically, let me say what it means. An operator is an orthogonal projector.

The test is that the operator squared is equal to itself. That's the name of a projector. And the other thing is that it must be Hermitian. These two things must happen for something to be an orthogonal projector.

Basically, what does it mean to be an orthogonal projector? It's an operator that projects to some space. So an operator has a set of states that are in its image in the range of the operator. That's what it projects into.

And it has a null space-- the things that are killed by the projector. A projector has precisely that kind of thing. So the thing that makes it an orthogonal projector is that the range of the projector-- the vectors that you get-- are orthogonal to the vectors that are killed by the projector.

The whole vector space splits into two parts-- what do you get from the projector and the rest. And the rest is things that are killed by the projector. And they're orthogonal to it.

So let me say it here. A projector P sub U to U subset of V is orthogonal-- so this is 805 type stuff-- if the vector space can be written as the null space of PU. Plus the range of PU with null or PU perpendicular, or orthogonal, to range of PU.

So this is the definition of an orthogonal projector. And here is, of course, a simple claim. A Hermitian operator, P such that P squared is equal to P, is an orthogonal projector. Projector. OK. So that's second claim.

So let's see again. What we want to show is that these are projectors, and they're orthogonal projectors. And this is what it means. But practically speaking, all we have to do is check that this operator satisfies this and our Hermitian. That's all you need to do.

Let me explain why this is the case. So whenever you have-- let's do a little proof. So if you have a vector, a vector can be written as the projector times the vector plus 1 minus the projector acting on the vector.

This-- the projector on the vector-- is in the range of P, because the range of P are the vectors obtained by acting with P. Moreover, this is in the null of P. Why? Because P kills these vectors.

How do you check that? P, acting on that vector-- P with this is P. But P squared is equal to P. So you get 0.

So here it is. The vector space decomposes into the range of P plus the null of P. So that's the first claim for a orthogonal projector.

The second claim is that these are orthogonal. So if I take PV-- something in the range of P-- and 1 minus PV tilde-- something in the null space of P-- this should be orthogonal. And this is clear, because I can move now this operator to the other side. And it will be V P dagger 1 minus P V tilde.

And since P is Hermitian-- that's where you use that P is Hermitian-- this is V P1 minus P V tilde. And that's equal to 0, because P times 1 minus P already is 0. So OK. These are our conditions.

Orthogonal projectors are very nice. These are the really good things. You have a projector that decomposes into two parts, just by having P squared equal P.

But to have orthogonal, you need that they'd be Hermitian. So what is our situation with those operators? They are Hermitian.

So S and A are Hermitian, because one is Hermitian, and P2,1 is Hermitian. So the operators are Hermitian. And they are projectors, because, well, you check them. You multiply them by themselves.

So we can do the two of them simultaneously. This is with a plus sign. That's S. With a minus sign, it's A.

So we need to show that S squared is equal to S and A squared is equal to A. So here we have both cases. So the product is 1/4, then plus minus 2 P2,1, and then the product of these operators with themselves. Plus minus and plus minus always gives you a plus.

And P2,1 times P2,1, we showed that that's equal to 1. So happily, this 1 and 1 give a 2. There's another 2. Divides here-- this is 1/2 1 plus minus P2,1. So that's what we wanted to show with a top sign, S times S is equal to S, with a bottom sign, A times A is equal to A.