Quantum Physics III (8.06) — Spring 2018

Assignment 3

Posted: Wednesday, February 21, 2018

Announcements

- Please put **your name and section** at the top of what you hand in.

Readings

- Griffiths, Section 6.3, 6.4, and 6.5.
- Cohen-Tannoudji, Chapter XII
- Shankar, Chapter 17.

1. **Griffiths 6.15, p.270 (15 points)** Here is the problem statement with minor modifications! (Beware this was only corrected in the Cambridge University Press Second Edition (2016))

   (a) Show that p^2 is hermitian for hydrogen states with $\ell = 0$. For such states ψ is independent of θ and ϕ so

   $$p^2 \equiv -\frac{\hbar^2}{r^2}\frac{d}{dr}\Big(r^2\frac{d}{dr}\Big)$$

   Using integration by parts, show that Hermiticity works up to a boundary term

   $$\langle f|p^2 g\rangle \;=\; -4\pi\hbar^2\Big(r^2 f\frac{dg}{dr} - r^2 g\frac{df}{dr}\Big)\Big|_0^\infty + \langle p^2 f|g\rangle\,.$$

   Consider the wavefunction for a state ψ_{n00}, which goes like

   $$\psi_{n00}(r) \;\sim\; N\,(1 + c_1 r + \ldots + c_{n-1}r^{n-1})\,\exp\Big(-\frac{r}{na_0}\Big)\,,$$

   where N and the c_i's are constants. Check that the boundary term vanishes when $g = \psi_{n00}$ and $f = \psi_{n'00}$.

   (b) The case of p^4 is more subtle. The laplacian of $1/r$ picks up a delta function. Show that

   $$\nabla^2\nabla^2 e^{-kr} \;=\; \Big(-\frac{4k^3}{r} + k^4\Big)e^{-kr} + 8\pi k\,\delta^3(\mathbf{r})\,.$$

   Use this relation to verify that p^4 is Hermitian when evaluating the inner product $\langle e^{-k'r}|p^4 e^{-kr}\rangle$.

2. **Wavefunction at the origin for spherically symmetric eigenstates (15 points)** (based on Sakurai's Modern Quantum Mechanics)

   For a particle in a zero angular momentum ($\ell = 0$) bound state of a central potential $V(r)$ a rather surprising result relates the value of the wavefunction $\psi(0)$ at the origin to the expectation value of a derivative of the potential:

   $$|\psi(0)|^2 \sim \left\langle \frac{dV}{dr} \right\rangle .$$

   Derive such relation and fix the coefficient precisely. (*Hints:* Begin with the radial equation for $u(r)$, multiply the equation by $u'(r)$ and integrate the equation from $r = 0$ to $r = \infty$.)

   Use the result to calculate $|\psi_{n00}(0)|^2$ for the nS states of the hydrogen atom. Verify you got the right answer for $n = 1$.

3. **Numerical estimates (5 points)**

   (i) Find the value of an external magnetic field that acting on a free electron produces energy levels that have a separation equal to the splitting between $3P_{1/2}$ and $3P_{3/2}$ states.

   (ii) Estimate the internal magnetic field at the electron in the 3P states. (You may use Griffiths eqns. (6.59) and (6.64)).

4. **Hydrogen medley (25 points)** Let m_e denote the mass of an electron and e its charge. The 8.04 version of the Hydrogen Hamiltonian is

   $$H_0 = \frac{p^2}{2m} - \frac{e^2}{r}.$$

   In this problem we will consider the spin of the electron (whose corresponding operator we call $\vec{S}$) but we will ignore the spin of the proton.

   (a) **Complete sets of commuting observables (CSCO).** A CSCO is a set of commuting operators whose simultaneous eigenspaces are each one dimensional. Equivalently, specifying all the eigenvalues of all the operators in a CSCO uniquely specifies a state (up to multiplication by a scalar). You may use without proof that i) $H_0, \vec{L}^2, L_z, S_z$ form a CSCO with eigenbasis $\{|n, l, m_l, m_s\rangle\}$, and ii) $H_0, \vec{L}^2, \vec{J}^2, J_z$ is a CSCO with eigenbasis $\{|n, l, j, m_j\rangle\}$. (In fact, using the rules for addition of angular momentum, ii) follows from i)). For each of the following sets of operators, either (a) explain why they are CSCOs or (b) explain why they are *not* CSCOs. For simplicity, consider only bound states.

      i. $H_0, \vec{L}^2, \vec{L} \cdot \vec{S}, J_z$.
      ii. $H_0, \vec{L}^2, L_z, S_x$.
      iii. $H_0, \vec{L}^2, J_z, S_z$.
      iv. $H_0, \vec{J}^2, J_z, S_z$.
      v. $H_0, \vec{J}^2, \vec{L} \cdot \vec{S}, J_z$.

(b) **Strong-field Zeeman effect.** In the strong-field Zeeman effect, the unperturbed eigenstates are the uncoupled states $|n, l, m_l, m_s\rangle$ whose energies have been shifted by an amount proportional to $m_l + 2m_s$ due to the Zeeman Hamiltonian. The more challenging part of the computation is to take care of the fine structure, which can be thought of as contributing a term

$$H_{\rm fs} = -\frac{m_e c^2 \alpha^4}{2n^3}\left(\frac{1}{\hat{j}+1/2} - \frac{3}{4n}\right),$$

where $\hat{j}$ is an operator satisfying $J^2 = \hbar^2 \hat{j}(\hat{j}+1)$. To compute the first-order energy shifts here we need to evaluate

$$\langle n, l, m_l, m_s|\frac{1}{\hat{j}+1/2}|n, l, m_l, m_s\rangle. \tag{1}$$

i. Use the following strategy to evaluate (1). First compute the expectation value of J^2 on the $|n, l, m_l, m_s\rangle$ state. Now imagine that we measure $\hat{j}$. Use your calculation to find the probabilities of the two outcomes $j = l + 1/2$ and $j = l - 1/2$. Finally use the fact that(1) equals

$$\frac{\Pr[j = l+1/2]}{l+1} + \frac{\Pr[j = l-1/2]}{l} \tag{2}$$

to reproduce the known result for E^1_{fs} in the strong-field Zeeman effect (see Griffiths (6.82)). The full Zeeman effect includes the contribution from the external magnetic field.

ii. The shifts in (1) used non-degenerate perturbation theory. But even with the unperturbed energies a function of $m_l + 2m_s$ there are still degeneracies (it may help you understand things better if you display those degeneracies for the $n = 2$ level). Explain carefully and in detail why, despite these degeneracies, the above argument is still correct.

5. **Identities with vector operators.** (20 points)

Suppose you have a set of angular momentum operators $\hat{J}_i$, $i = 1, 2, 3$ that define an angular momentum $\hat{\mathbf{J}}$. A set of operators $\hat{W}_i$, with $i = 1, 2, 3$ is said to form a **vector operator $\hat{\mathbf{W}}$** if

$$[\,\hat{J}_i\,, \hat{W}_j] \;=\; i\hbar\,\epsilon_{ijk}\,\hat{W}_k\,.$$

Note that $\hat{\mathbf{J}}$ itself is a vector operator.

(a) Show that when $\hat{\mathbf{J}}$ is taken to be the orbital angular momentum $\hat{\mathbf{L}}$ the position operator $\hat{\mathbf{x}}$ is a vector operator.

(b) Show that if $\hat{\mathbf{U}}$ and $\hat{\mathbf{V}}$ are vector operators, so is the cross product $\hat{\mathbf{U}} \times \hat{\mathbf{V}}$.

(c) Show that if $\hat{\mathbf{W}}$ is a vector operator then

$$[\hat{\mathbf{J}}^2\,, \hat{\mathbf{W}}] \;=\; 2i\hbar\left(\hat{\mathbf{W}} \times \hat{\mathbf{J}} \;-\; i\hbar\,\hat{\mathbf{W}}\right)$$

Check that this formula holds when we choose $\hat{\mathbf{W}} = \hat{\mathbf{J}}$.

(d) Show that for $\hat{\mathbf{V}}$ a vector operator the following formula holds

$$\frac{1}{\alpha}\Big[\hat{\mathbf{J}}^2\,,[\hat{\mathbf{J}}^2\,,\hat{\mathbf{V}}]\Big] \;=\; (\hat{\mathbf{V}}\cdot\hat{\mathbf{J}})\,\hat{\mathbf{J}} \;-\; \tfrac{1}{2}\big(\hat{\mathbf{J}}^{\mathbf{2}}\,\hat{\mathbf{V}}+\hat{\mathbf{V}}\,\hat{\mathbf{J}}^{\mathbf{2}}\big)\,,$$

with α a constant you must determine.