

Quantum Physics III (8.06) Spring 2005

Assignment 4

Feb 22, 2005

Due March 1, 2005, 6pm

Readings

The reading assignment for this week was already given. You should focus on completing it.

If you wish to read ahead, the next reading assignment will be:

- Griffiths Sections 6.1, 6.2
- Cohen-Tannoudji Chapter XI

Problem Set 4

1. Off-Diagonal Conductance in Two-Dimensions (10 points)

Consider a two-dimensional strip of material of length L and width W , placed in a magnetic field perpendicular to the strip and with an electric field established in the direction of the length L .

(a) Suppose that the resistivity matrix is given by the classical result

$$\rho = \begin{pmatrix} \rho_0 & -\rho_H \\ \rho_H & \rho_0 \end{pmatrix} \quad (1)$$

where $\rho_H = B/nec$ is the Hall resistivity and ρ_0 is the usual Ohmic resistivity. Find the conductivity matrix, $\sigma = \rho^{-1}$. Write it in the form:

$$\sigma = \begin{pmatrix} \sigma_0 & \sigma_H \\ -\sigma_H & \sigma_0 \end{pmatrix}. \quad (2)$$

What are σ_0 and σ_H ?

(b) Suppose $B = 0$, so the Hall resistivity is zero. Notice that the Ohmic conductivity, σ_0 , is just $1/\rho_0$. In particular, note that $\sigma_0 \rightarrow \infty$ as $\rho_0 \rightarrow 0$. Now suppose $\rho_H \neq 0$. Show that $\sigma_0 \rightarrow 0$ as $\rho_0 \rightarrow 0$, so it is possible to have *both* σ_0 and ρ_0 equal to zero

- (c) Now consider the case where $\sigma_0 = \rho_0 = 0$. Show that the Hall *resistance* of the sample is equal to the Hall resistivity, and is therefore independent of L and W . (You should begin by showing that in two dimensions, unlike in three dimensions, resistance has the same dimensions as resistivity. This alone is not enough to argue that the resistance and resistivity are equal. Dimensional analysis alone does not preclude the occurrence of factors like L/W in the resistance. Indeed, if there were “on-diagonal” contributions to σ , you would find that the resistances do depend on L and W .)

2. Quantum States in Crossed \vec{E} and \vec{B} Fields (10 points)

This problem asks you to give a complete presentation of a calculation that is almost the same as one you saw in lecture.

Consider a constant electric field, $\vec{E} = (0, E_0, 0)$ and a constant magnetic field, $\vec{B} = (0, 0, B_0)$.

- (a) Choose an electrostatic potential ϕ and a vector potential \vec{A} which describe the \vec{E} and \vec{B} fields, and write the Hamiltonian for a charged particle of mass m and charge q in these fields. Assume that the particle is restricted to move in the xy -plane.
- (b) What are the allowed energies as a function of B_0 and E_0 ? Draw a figure to show how the Landau levels (energy levels when $E_0 = 0$) change as E_0 increases.

3. The Aharonov-Bohm Effect on Energy Eigenvalues (20 points)

You will see the “standard presentation” of the Aharonov-Bohm effect in lecture, on the day that this problem set is due. The standard presentation has its advantages, and in particular is more general than the presentation you will work through in this problem. However, students often come away from the standard presentation of the Aharonov-Bohm effect thinking that the only way to detect this effect is to do an interference experiment. This is not true, and the purpose of this problem is to disabuse you of this misimpression before you form it.

As Griffiths explains on pages 385-387 (344-345 in 1st Ed.), the Aharonov-Bohm effect modifies the energy eigenvalues of suitably chosen quantum mechanical systems. In this problem, I ask you to work through the same physical example that Griffiths uses, but in a different fashion which makes more use of gauge invariance.

Imagine a particle constrained to move on a circle of radius b (a bead on a wire ring, if you like.) Along the axis of the circle runs a solenoid of radius $a < b$, carrying a magnetic field $\vec{B} = (0, 0, B_0)$. The field inside the solenoid is uniform

and the field outside the solenoid is zero. The setup is depicted in Griffiths' Fig. 10.10. (10.12 in 1st Ed.)

- (a) Construct a vector potential \vec{A} which describes the magnetic field (both inside and outside the solenoid) and which has the form $A_r = A_z = 0$ and $A_\phi = \alpha(r)$ for some function $\alpha(r)$. I am using cylindrical coordinates z, r, ϕ .
- (b) Since $\vec{\nabla} \times \vec{A} = 0$ for $r > a$, it must be possible to write $\vec{A} = \vec{\nabla} f$ in any *simply connected* region in $r > a$. [This is a theorem in vector calculus.] Show that if we find such an f in the region

$$r > a \text{ and } -\pi + \epsilon < \phi < \pi - \epsilon ,$$

then

$$f(r, \pi - \epsilon) - f(r, -\pi + \epsilon) \rightarrow \Phi \text{ as } \epsilon \rightarrow 0 .$$

Here, the total magnetic flux is $\Phi = \pi a^2 B_0$. Now find an explicit form for f , which is a function only of ϕ .

- (c) Now consider the motion of a “bead on a ring”: write the Schrödinger equation for the particle constrained to move on the circle $r = b$, using the \vec{A} you found in (a). Hint: the answer is given in Griffiths.
- (d) Use the $f(\phi)$ found in (b) to gauge transform the Schrödinger equation for $\psi(\phi)$ within the angular region $-\pi + \epsilon < \phi < \pi - \epsilon$ to a Schrödinger equation for a *free* particle within this angular region. Call the original wave function $\psi(\phi)$ and the gauge-transformed wave function $\psi'(\phi)$.
- (e) The original wave function ψ must be single-valued for all ϕ , in particular at $\phi = \pi$. That is, $\psi(\pi - \epsilon) - \psi(-\pi + \epsilon) \rightarrow 0$ and $\frac{\partial \psi}{\partial \phi}(\pi - \epsilon) - \frac{\partial \psi}{\partial \phi}(-\pi + \epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. What does this say about the gauge-transformed wave function? I.e., how must $\psi'(\pi - \epsilon)$ and $\psi'(-\pi + \epsilon)$ be related as $\epsilon \rightarrow 0$?
[Hint: because the $f(\phi)$ is not single valued at $\phi = \pi$, the gauge transformed wave function $\psi'(\phi)$ is not single valued there either.]
- (f) Solve the Schrödinger equation for ψ' , and find energy eigenstates which satisfy the boundary conditions you derived in (e). What are the allowed energy eigenvalues?
- (g) Undo the gauge transformation, and find the energy eigenstates $\psi(\phi)$ in the original gauge. Do the energy eigenvalues in the two gauges differ?
- (h) Plot the energy eigenvalues as a function of the enclosed flux, Φ . Show that the energy eigenvalues are periodic functions of Φ with period Φ_0 , where you must determine Φ_0 . For what values of Φ does the enclosed magnetic field have no effect on the spectrum of a particle on a ring? Show that the

Aharonov-Bohm effect can only be used to determine the fractional part of Φ/Φ_0 .

[Note: you have shown that even though the bead on a ring is everywhere in a region in which $\vec{B} = 0$, the presence of a nonzero \vec{A} affects the energy eigenvalue spectrum. However, the effect on the energy eigenvalues is determined by Φ , and is therefore gauge invariant. To confirm the gauge invariance of your result, you can compare your answer for the energy eigenvalues to Griffiths' result, obtained using a different gauge.]

4. Perturbation of the Three-Dimensional Harmonic Oscillator (20 points)

The spectrum of the three-dimensional harmonic oscillator has a high degree of degeneracy. In this problem, we see how the addition of a perturbation to the Hamiltonian reduces the degeneracy. This problem is posed in such a way that you can work through it before we even begin to discuss perturbation theory in lecture.

Consider a quantum system described by the Hamiltonian

$$H = H_0 + H_1 \quad (3)$$

where

$$H_0 = \frac{1}{2m}\vec{p}^2 + \frac{1}{2}m\omega^2\vec{x}^2 \quad (4)$$

where $\vec{x} = (x_1, x_2, x_3)$ and $\vec{p} = (p_1, p_2, p_3)$. The perturbing Hamiltonian H_1 is given by

$$H_1 = KL_2 \quad (5)$$

where K is a constant and where $L_2 = x_3p_1 - x_1p_3$.

In parts (a)-(e) of this problem, we study the effects of this perturbation within the degenerate subspace of states which have energy $E = (5/2)\hbar\omega$ when $K = 0$.

- (a) Set $K = 0$. Thus, in this part of the problem $H = H_0$. Define creation and annihilation operators for "oscillator quanta" in the 1, 2 and 3 directions. Define number operators N_1, N_2, N_3 . Denote eigenstates of these number operators by their eigenvalues, as $|n_1, n_2, n_3\rangle$. What is the energy of the state $|n_1, n_2, n_3\rangle$? How many linearly independent states are there with energy $E = (5/2)\hbar\omega$? [That is, what is the degeneracy of the degenerate subspace of states we are studying?]
- (b) Express the perturbing Hamiltonian H_1 in terms of creation and annihilation operators.
- (c) What is the matrix representation of H_1 in the degenerate subspace you described in part (a)?

- (d) What are the eigenvalues and eigenstates of H_1 in the degenerate subspace? What are the eigenvalues and eigenstates of $H = H_0 + H_1$ in the degenerate subspace?
- (e) What is the matrix representation of $H_0 + H_1$ in the degenerate subspace if you use the eigenvectors of H_1 as a new basis? (I.e. instead of the original $|n_1, n_2, n_3\rangle$ basis.)

[Note: As we shall see in part (f), this problem is “too simple” in important ways. The aspect of this problem which *will* generalize when we consider more generic perturbations is that if a perturbation breaks a degeneracy, then even an arbitrarily small but nonzero perturbation has qualitative consequences: it selects one particular choice of energy eigenvectors, within the previously degenerate subspace. In the present problem, this can be described as follows: if K were initially zero and you were happily using the $|n_1, n_2, n_3\rangle$ states as your basis of energy eigenstates, and then somebody “turns on” a very small but nonzero value of K , this forces you to make a qualitative change in your basis states. The “rotation” you must make from your previous energy eigenstates to the new states which are now the only possible choice of energy eigenstates is not a small one, even though K is arbitrarily small.]

- (f) $|\psi\rangle$ and $|\phi\rangle$ are eigenstates of H_0 with *different* energy eigenvalues. That is, $|\psi\rangle$ and $|\phi\rangle$ belong to different degenerate subspaces. Show that $\langle\phi|H_1|\psi\rangle = 0$ for any two such states. Relate this fact to a statement you can make about the operators H_0 and H_1 , without reference to states.

[The fact that $\langle\phi|H_1|\psi\rangle = 0$ if $|\psi\rangle$ and $|\phi\rangle$ belong to different degenerate subspaces means that H_1 is a “non-generic” perturbation of H_0 ; a more general perturbation would not have this property. It is only for perturbations with this property that the analysis you have done above — which focusses on one degenerate subspace at a time — is complete. Notice also that in order to analyze $H = H_0 + H_1$, we did not have to assume that K was in any sense small. If H_1 were “generic”, we would have had to assume that K was small in order to make progress.]