8.05, Quantum Physics II, Fall 2012
TEST
Wednesday October 24, 12:00-1:30pm
You have 90 minutes.

Answer all problems in the white books provided. Write YOUR NAME and YOUR SECTION on your white book(s).

There are five questions, totalling 100 points.

None of the problems requires extensive algebra.

No books, notes, or calculators allowed.

TIME MANAGEMENT: I suggest 40 minutes for the first three questions and 25 minutes for each of the remaining two.

Formula Sheet

- Conservation of probability

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}J(x,t) = 0$$

$$\rho(x,t) = |\psi(x,t)|^2 \; ; \qquad J(x,t) = \frac{\hbar}{2im}\left[\psi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right]$$

- Variational principle:

$$E_{gs} \leq \frac{\int dx\, \psi^*(x) H \psi(x)}{\int dx \psi^*(x)\psi(x)}\,, \quad \text{for all } \psi(x)$$

- Spin-1/2 particle:

$$\text{Stern-Gerlach :} \quad H = -\vec{\mu}\cdot\vec{B}\,, \qquad \vec{\mu} = g\;\frac{e\hbar}{2m}\;\frac{1}{\hbar}\vec{S} = \gamma\vec{S}$$

$$\mu_{_B} = \frac{e\hbar}{2m_e}\,, \qquad \vec{\mu}_e = -2\,\mu_{_B}\frac{\vec{S}}{\hbar}\,,$$

$$\text{In the basis } |1\rangle \equiv |z;+\rangle = |+\rangle = \begin{pmatrix}1\\0\end{pmatrix}\,,\; |2\rangle \equiv |z;-\rangle = |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$S_i = \frac{\hbar}{2}\sigma_i \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\;;\; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\;;\; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad \big[S_i\,,\,S_j\big] = i\hbar\,\epsilon_{ijk}S_k \quad (\epsilon_{123} = +1)$$

$$\sigma_i\sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b}\,I + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

$$e^{i\mathbf{M}\theta} = \mathbf{1}\cos\theta + i\mathbf{M}\sin\theta\,, \quad \text{if } \mathbf{M}^2 = \mathbf{1}$$

$$\exp\big(i\vec{a}\cdot\vec{\sigma}\big) = \mathbf{1}\cos a + i\vec{\sigma}\cdot\Big(\frac{\vec{a}}{a}\Big)\,\sin a\,, \quad a = |\vec{a}|$$

$$\begin{aligned}\exp(i\theta\sigma_3)\,\sigma_1\,\exp(-i\theta\sigma_3) &= \sigma_1\cos(2\theta) - \sigma_2\sin(2\theta)\\ \exp(i\theta\sigma_3)\,\sigma_2\,\exp(-i\theta\sigma_3) &= \sigma_2\cos(2\theta) + \sigma_1\sin(2\theta)\;.\end{aligned}$$

$$S_{\vec{n}} = \vec{n}\cdot\vec{S} = n_xS_x + n_yS_y + n_zS_z = \frac{\hbar}{2}\,\vec{n}\cdot\vec{\sigma}\;.$$

$$(n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \quad S_{\vec{n}}\,|\vec{n};\pm\rangle = \pm\frac{\hbar}{2}\,|\vec{n};\pm\rangle$$

$$\begin{aligned}|\vec{n};+\rangle &= \cos(\theta/2)|+\rangle + \sin(\theta/2)\exp(i\phi)|-\rangle\\ |\vec{n};-\rangle &= -\sin(\theta/2)\exp(-i\phi)|+\rangle + \cos(\theta/2)|-\rangle\end{aligned}$$

- Bras and kets: For an operator Ω and a vector v, we write $|\Omega v\rangle \equiv \Omega|v\rangle$

$$\text{Adjoint:} \quad \langle u|\Omega^\dagger v\rangle \; = \; \langle \Omega u|v\rangle$$

$$|\alpha_1 v_1 + \alpha_2 v_2\rangle = \alpha_1|v_1\rangle + \alpha_2|v_2\rangle \quad \longleftrightarrow \quad \langle \alpha_1 v_1 + \alpha_2 v_2| = \alpha_1^*\langle v_1| + \alpha_2^*\langle v_2|$$

- Complete orthonormal basis $|i\rangle$

$$\langle i|j\rangle = \delta_{ij}\,, \qquad \mathbf{1} = \sum_i |i\rangle\langle i|$$

$$\Omega_{ij} = \langle i|\Omega|j\rangle \quad \leftrightarrow \quad \Omega = \sum_{i,j} \Omega_{ij}\; |i\rangle\langle j|$$

$$\langle i|\Omega^\dagger|j\rangle = \langle j|\Omega|i\rangle^*$$

$$\Omega \;\; \text{hermitian:} \;\; \Omega^\dagger = \Omega, \qquad U \; \text{unitary:} \;\; U^\dagger = U^{-1}$$

- Matrix M is normal ($[M, M^\dagger] = 0$) $\longleftrightarrow$ unitarily diagonalizable.

- Position and momentum representations: $\psi(x) = \langle x|\psi\rangle$; $\tilde{\psi}(p) = \langle p|\psi\rangle$;

$$\hat{x}|x\rangle = x|x\rangle\,, \quad \langle x|y\rangle = \delta(x-y)\,, \quad \mathbf{1} = \int dx\; |x\rangle\langle x|\,, \quad \hat{x}^\dagger = \hat{x}$$

$$\hat{p}|p\rangle = p|p\rangle\,, \quad \langle q|p\rangle = \delta(q-p)\,, \quad \mathbf{1} = \int dp\; |p\rangle\langle p|\,, \quad \hat{p}^\dagger = \hat{p}$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\big(\frac{ipx}{\hbar}\big)\,; \qquad \tilde{\psi}(p) = \int dx \langle p|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx \exp\big(-\frac{ipx}{\hbar}\big)\psi(x)$$

$$\langle x|\hat{p}^n|\psi\rangle = \Big(\frac{\hbar}{i}\frac{d}{dx}\Big)^n \psi(x)\,; \qquad \langle p|\hat{x}^n|\psi\rangle = \Big(i\hbar\frac{d}{dp}\Big)^n \tilde{\psi}(p)\,; \qquad [\hat{p}, f(\hat{x})] = \frac{\hbar}{i} f'(\hat{x})$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ikx)dx = \delta(k)$$

- Generalized uncertainty principle

$$(\Delta A)^2 \equiv \langle (A - \langle A\rangle)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$$

$$(\Delta A)^2(\Delta B)^2 \geq \Big(\langle\Psi|\frac{1}{2i}[A, B]|\Psi\rangle\Big)^2$$

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}$$

$$\Delta x = \frac{\Delta}{\sqrt{2}} \;\; \text{and} \;\; \Delta p = \frac{\hbar}{\sqrt{2}\Delta} \;\; \text{for a gaussian wavefuntion } \psi \sim \exp\Big(-\frac{1}{2}\frac{x^2}{\Delta^2}\Big)$$

$$\int_{-\infty}^{+\infty} dx \exp\left(-ax^2\right) = \sqrt{\frac{\pi}{a}}$$

$$\text{Time independent operator } Q: \quad \frac{d}{dt}\langle Q\rangle \;=\; \frac{i}{\hbar}\langle\,[H,Q]\rangle$$

$$\Delta H \Delta t \geq \frac{\hbar}{2}\,, \quad \Delta t \equiv \frac{\Delta Q}{\left|\frac{d\langle Q\rangle}{dt}\right|}$$

- Commutator identities

$$[A, BC] = [A,B]C + B[A,C]\,,$$

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \ldots\,,$$

$$e^A B e^{-A} = B + [A,B]\,, \quad \text{if} \quad [[A,B],A] = 0\,,$$

$$[\,B\,,\,e^A\,] = [\,B\,,A\,]e^A\,, \qquad \text{if} \quad [[A,B],A] = 0$$

$$e^{A+B} \;=\; e^A e^B e^{-\frac{1}{2}[A,B]} \;=\; e^B e^A e^{\frac{1}{2}[A,B]}\,, \quad \text{if } [A,B] \text{ commutes with } A \text{ and with } B$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)\,, \quad \hat{N} = \hat{a}^\dagger\hat{a}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\,, \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\,,$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)\,, \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})\,,$$

$$[\hat{x},\hat{p}] = i\hbar\,, \quad [\hat{a},\hat{a}^\dagger] = 1\,, \qquad [\hat{N},\hat{a}\,] = -\hat{a}\,, \quad [\hat{N}\,,\hat{a}^\dagger\,] \;=\; \hat{a}^\dagger\,.$$

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$

$$\hat{H}|n\rangle = E_n|n\rangle = \hbar\omega\big(n + \frac{1}{2}\big)|n\rangle\,, \quad \hat{N}|n\rangle = n|n\rangle\,, \quad \langle m|n\rangle = \delta_{mn}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\,, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle\,.$$

$$\psi_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\,.$$

1. **True or false questions** [20 points] No explanations required. Just indicate T or F for true or false, respectively.

   (1) The parity operator, $\hat{P}|x\rangle = |-x\rangle$, is both hermitian and unitary.

   (2) For position eigenstates: $|-x\rangle = -|x\rangle$

   (3) Every $N \times N$ matrix has N eigenvectors.

   (4) For a finite dimensional Hilbert space with N energy levels $E_1 \leq E_2 \leq \ldots \leq E_N$, $\langle\psi|H|\psi\rangle \leq E_N$ for all normalized trial wavefunctions $|\psi\rangle$.

   The following three questions refer to a one-dimensional $x \in (\infty, \infty)$ Schrödinger problem with a real, continuous and finite potential $V(x) = V(-x)$ that admits five bound states.

   (5) The highest energy bound state has four nodes.

   (6) Let $\psi(x)$ denote a bound state wavefunction. There may exist points x_0 ($\neq \infty$) where both ψ and is spatial derivative ψ' vanish.

   (7) An excited even bound state can have $\psi(x=0) = 0$.

   The following three questions refer to the operator

$$R = \exp\Big(\frac{i}{\hbar}\pi\hat{S}_x\Big)$$

   where $\hat{S}_x$ is the spin operator in the x direction.

   (8) The operator R is unitary.

   (9) The operator R is diagonalizable.

   (10) $R \ = \frac{2i}{\hbar}\hat{S}_x$.

2. **Expectation values and uncertainty** [15 points]

   A useful equation relates the action of a Hermitian operator Ω on a normalized state $|\Psi\rangle$ to its expectation value $\langle\Omega\rangle$ and its non-zero uncertainty $\Delta\Omega$ in the state:

$$\Omega|\Psi\rangle \ = \ \langle\Omega\rangle|\Psi\rangle + (\Delta\Omega)|\Psi_\perp\rangle\,. \qquad (1)$$

   Here $|\Psi_\perp\rangle$ is (i) a normalized state, and (ii) is orthogonal to $|\Psi\rangle$. Prove the above equation by showing that the state $|\Psi_\perp\rangle$ defined by (1) has the two properties we claim it has.

3. **A property of complex vector spaces** [15 points]

   Consider two vectors $|u\rangle$ and $|v\rangle$ in a complex vector space $\mathbb{V}$ as well as the linear operator $T : \mathbb{V} \to \mathbb{V}$.

   (a) Simplify the following expression

   $$\frac{1}{4}\Big[\ \langle u+v|T|u+v\rangle\ -\ \langle u-v|T|u-v\rangle - i\,\langle u+iv|T|u+iv\rangle\ +i\,\langle u-iv|T|u-iv\rangle\ \Big]$$

   and write your answer in terms of (some or all of) the overlaps $\langle u|T|u\rangle$, $\langle u|T|v\rangle$, $\langle v|T|u\rangle$, and $\langle v|T|v\rangle$.

   (b) Use your result in (a) to prove that if $\langle w|T|w\rangle = 0$ for all $w \in \mathbb{V}$ then T is the zero operator.

   [Comment: This is a remarkable property of complex vector spaces that is *not* true in real vector spaces. For a two-dimensional real vector space let T be the linear operator that rotates vectors by $90°$. Then T is non-zero even though $\langle v|T|v\rangle = 0$.]

4. **An Anharmonic Oscillator** [25 points]

   Consider a particle of mass m moving in one-dimension under the influence of an x^{2n} potential

   $$V(x) \;=\; \frac{\hbar^2}{2m}\,\frac{x^{2n}}{L^{2n+2}}\,, \qquad n \geq 2\,,$$

   with L a constant with units of length.

   (a) Use dimensional analysis to estimate the ground state energy of the system up to an undetermined dimensionless number.

   (b) Consider the trial wavefunction $\psi(x) = \exp\bigl(-b^2\,x^2\bigr)$ for a variational analysis of the ground with b a parameter to be adjusted to obtain the best bound. Consider also the following integrals

   $$\int_{-\infty}^{\infty} dy\, e^{-2y^2} y^{2k} = c_k\,, \quad k = 0, 1, 2, \ldots$$

   and assume the constants c_k known. Determine the function $F(b)$ that bounds the ground state energy as

   $$E_{gs} \leq F(b)$$

   You do not have to minimize over b, but simplify your result for F, which also depends on other constants of the problem.

   (c) Sketch the potential in the limit of $n \to \infty$. What do you expect the ground state energy to be? Explain.

   (d) One can show (don't try!) that the value of b for best variational estimate in part (b) goes like $bL \sim \sqrt{n}$ for large n. Sketch the expected ψ and explain if this is becoming a better or worse representation of the expected ground state as n becomes larger and larger.

5. **A Three Dimensional Harmonic Oscillator** [25 points]

Consider a particle of mass m confined to a three-dimensional harmonic potential with rotational symmetry:

$$\hat{H} = \sum_{i=1}^{3} \Big(\frac{\hat{p}_i^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}_i^2 \Big),$$

where $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$. The corresponding raising and lowering operators are denoted as $\hat{a}_i^\dagger$ and $\hat{a}_i$, with $i = 1, 2, 3$, and we have number operators $\hat{N}_i = \hat{a}_i^\dagger \hat{a}_i$, also with $i = 1, 2, 3$. (You are welcome to drop the "hats" from the a's and $a^\dagger$ to save time!)

(a) Using the $\hat{a}_i^\dagger$ operators and a common vacuum state $|0\rangle$ for the three $\hat{a}_i$'s, explicitly construct all states in (i) the ground state, (ii) the first excited level, and (iii) the second excited level. Give their energies and state the degeneracy (The states need not be normalized.)

We introduce the useful linear combinations

$$\hat{a}_L \equiv \frac{1}{\sqrt{2}}(\hat{a}_1 + i\hat{a}_2), \qquad \hat{a}_R \equiv \frac{1}{\sqrt{2}}(\hat{a}_1 - i\hat{a}_2).$$

(b) Express $\hat{H}$ in terms of the number operators

$$\hat{N}_L = \hat{a}_L^\dagger \hat{a}_L, \quad \hat{N}_R = \hat{a}_R^\dagger \hat{a}_R\,, \quad \hat{N}_3 = \hat{a}_3^\dagger \hat{a}_3\,.$$

Explain why $\{\hat{N}_L,\, \hat{N}_R,\, \hat{N}_3\}$ form a complete set of commuting observables. What is the energy of the state $|n_L, n_R, n_3\rangle$?

(c) Make the list of states of the ground state and first two excited levels (as in (a)) but using the $\hat{a}_L^\dagger$, $\hat{a}_R^\dagger$, and $\hat{a}_3^\dagger$ operators to build the states.

(d) Consider the angular momentum operator

$$\hat{L}_z \;\equiv\; \hbar\,(\hat{a}_R^\dagger \hat{a}_R - \hat{a}_L^\dagger \hat{a}_L) \;=\; \hbar(\hat{N}_R - \hat{N}_L)\,.$$

Show that $[\hat{H}, \hat{L}_z] = 0$. Do $\hat{H}$ and $\hat{L}_z$ form a complete set of commuting observables? If yes, give an argument. If not, give an example of two degenerate states that are not distinguished.

(e) Now consider the angular momentum operator $\hat{L}_+ \equiv \hat{L}_x + i\hat{L}_y$ and given by

$$\hat{L}_+ \;\equiv\; \sqrt{2}\,\hbar\,(\,\hat{a}_3^\dagger\,\hat{a}_L - \hat{a}_R^\dagger\,\hat{a}_3\,).$$

Is $\hat{L}_+$ Hermitian? From the second excited states of the oscillator, one can form a unique state $|\psi\rangle$ that is killed both by $\hat{L}_z$ and by $\hat{L}_+$ (i.e. $\hat{L}_z|\psi\rangle = 0$ and $\hat{L}_+|\psi\rangle = 0$). Find it. This is the unique state of the second excited level that has no angular momentum whatsoever!