

PROFESSOR: This was a differential equation for the energy eigenstates ψ . Supposed to be normalizable functions. We looked at this equation and decided we would first clean out the constant. We did that by replacing x by a unit-free coordinate u .

For that we needed a constant that carries units of length, and that constant is given by this combination of the constants of the problem. \hbar , m , and ω -- the frequency of the oscillator.

We also defined the unit-free energy-- calligraphic e . In terms of which the real energies are given by multiples of $\hbar \omega$ over 2. So the problem has now become-- and this whole differential equation turns into this simple differential equation. Simple looking-- let's say properly-- differential equation for ψ -- as a function of u -- which is the new rescaled coordinate, and where the energy shows up here.

And for some reason, this equation doesn't have normalizable solutions. Unless those energies are peculiar values that allow a normalizable solution to exist.

We looked at this equation for u going to infinity, and realized that e to the plus minus u squared over 2 are the possible dependencies. So we said-- without loss of generality-- that ψ could be written as some function of u to be determined times this exponential.

And we hope for a function that may be a polynomial. So that the dependence at the infinity is governed by this factor.

So with this for the function ψ , we substitute back into the differential equation. Now the unknown is h . So you can take the derivatives and find this differential equation for h -- a second order differential equation. And that's the equation.

It may look a little more complicated than the equation we started with, but it's much simpler, actually. There would be no polynomial solution of this equation, but there may be a polynomial solution of the second equation.

So we have to solve this equation now. And the way to do it is to attempt a series expansion. So we would try to write h of u equals the sum over j . Equals 0 to infinity. $A_k u$ to the k . P equals 0 to infinity.

Now, one way to proceed with this is to plot this expansion into the differential equation. You

will get three sums. You will have to shift indices. It's kind of a little complicated. Actually, there's a simpler way to do this in which you think in the following way.

You have this series and you imagine there's a term $a_j u^j$, plus $a_{j+1} u^{j+1}$, plus $a_{j+2} u^{j+2}$. And you say, let me look at the terms with u^j in the differential equation. So just look at the terms that have a u to the power j .

So from this second-- $h v u^2$ -- what do we get? Well, to get a term that has a u to the j , you must start-- if you take two derivatives and to end up with u^j -- you must have started with this. u^{j+2} .

So this gives you $j+2, j+1$ -- taking the two derivatives-- $a_{j+2} u^j$. From the series, the term with u^j from the second $h v u^2$ is this one.

How about for $-2u \frac{dh}{du}$? Well, if I start with h and differentiate and then multiply by u , I'm going to get u^j starting from u^j . Because when I differentiate I'll get u^{j-1} , but the u will bring it back. So this time I get $-2 a_j$ or $-2 a_j u^j$. That's $a_j u^j$.

So it's from this [? step. ?] $-2 a_j$. You differentiate and you get that.

From the last term, $e^{-1} h$, it's clearly $e^{-1} a_j u^j$. So these are my three terms that we get from the differential equation.

So at the end of the day, what have we gotten? We've gotten $a_{j+2} u^j$, $a_{j+1} u^j$, $a_{j+2} u^j$, $-2 a_j u^j$, plus $e^{-1} a_j u^j$. All multiplied by u^j . And that's what we get for u^j .

So if you wish, for the whole differential equation-- all of the differential equation-- you get the sum from j equals 0 to infinity of these things. And that should be equal to zero.

So this is the whole left-hand side of the differential equation. We calculated what is the term u^j . And there will be terms from u^0 to u^∞ . So that's the whole thing.

And we need this differential equation to be solved. So this must be zero. And whenever you have a function of u like a polynomial-- well, we don't know if it's a polynomial-- and it stops. But if you have a function of u like this, each coefficient must be 0.

Therefore, we have that $j + 2$, times $j + 1$, times $a_j + 2$, is equal to $2j + 1$ minus e , a_j . I set this whole combination inside brackets to 0. So this term is equal to this term and that term on the other side. You get a_{j+2} . a_{j+1} and minus e .

So basically this is a recursion relation. a_{j+2} is equal to $2j + 1$ minus e , over $j + 2$, $j + 1$ times a_j .

And this is perfectly nice. This is what should have happened for this kind of differential equation-- a second-order linear differential equation. We get a recursion that jumps one step. That's very nice. And this should hold for j equals 0, 1, 2-- all numbers.

So when you start solving this, there's two ways to solve it. You can decide, OK, let me assume that you know a_0 -- you give it. Give a_0 . Well, from this equation-- from a_0 -- you can calculate a_2 . And then from a_2 you can calculate a_4 . And successively.

So you get a_2 , a_4 -- all of those. And this corresponds to an even solution of the differential equation for h . Even coefficients. Even solution for h . Or-- given this recursion-- you could also give a_1 -- give it-- and then calculate a_3 , a_5 -- and those would be an odd solution.

So you need two conditions to solve this. And those conditions are a_0 and a_1 , which is the same as specifying the value of the function h at 0-- because the value of the function h at 0 is a_0 . And the value of the derivative of the function at 0, which is a_1 .

[INAUDIBLE] h of u is a_0 plus $a_1 u$ plus $a_2 u$ squared. So the derivative at 0 is [h_1 . ?] And that's what you must have for solving a differential equation-- a second-order differential equation for h . You need to know the value of the function at zero and the value of the function at the derivative of the function-- at zero, as well.

And then you can start integrating it. So this first gives you a solution a_0 plus $a_2 u$ squared, plus $a_4 u$ to the fourth. And the second is an $a_1 u$ plus $a_3 u$ cubed, plus these ones.

So all looks pretty much OK.