PROFESSOR: Now we prove the other thing that we used in order to solve the square well. So this is property number 3. It's so important that I think I should do it here.

If a potential is even-- here comes again the careful statement-- the energy eigenstates can be chosen to be even or odd under x goes to minus x. So that's analog of the first sentence in property number 2.

But then comes the second sentence, that you can imagine what it is. For 1D potentials the bound states are either even or odd.

So look again at this freedom. You have a general problem-- you're not talking bound states. You have a wave function that solves a problem of a potential of the symmetric around a mid-point. Then you find an arbitrary solution, no need to work with that solution. You can work with a solution that is even, and a solution that is odd. You can always choose to be even or odd.

But if you have one-dimensional potential, there is no such solution that is neither even nor odd. You cannot find it. It will be, automatically, even or odd-- which is kind of remarkable. It's sufficiently subtle that in the general exam at MIT for graduate quantum mechanics, the professor that invented the problem forgot this property and the problem had to be cancelled.

It's a very interesting. So let's just try to prove it. So complete proof in this case. So what is the equation?

Proof. What is the equation we have to solve? Sine double prime of x, plus 2M over h squared. E minus v of x. Equals 0.

Now, the proof, actually, is very simple. I just do it and I elaborate on it, because it's possible to get a little confused about it. I think it's kind of interesting.

So here's equation one. And sine double prime of x notation means the second derivative of psi evaluated at x. You see, what you want to do is to show that psi of minus x solves the same equation. Right?

It's kind of clear. Well, you sort of put psi of minus x here, minus x here. Well, you would do minus x here. But if the potential is even it will solve the same equation. Now, the only complication here is that there are a few X's in the derivatives here. But whether there's a complication or not there's two derivatives, so the [? signs ?] should not matter.

But I want to make this a little clearer, and in order to do that I will just define phi of x to be equal to psi at minus x. So if you have that, the derivative of phi-- with respect to x-- you must differentiate this with respect to the argument. You evaluate at the argument, and then differentiate the argument with respect to x. So that leaves you a minus 1.

On the second derivative of phi with respect to x squared-- at x-- will be yet another derivative. So you now get a second derivative evaluated at the thing. And then differentiate the thing inside again. So minus 1 plus another minus 1. So this is just psi double prime of x-- of minus x.

Now, evaluate equation one at minus x. Well, it would be the second derivative of psi evaluated at minus x, plus 2m h squared, E minus V of minus x-- but that's the same as v of x-- psi at minus x equals 0.

And then you'll realize that this thing is just the second phi of x. [? h squared ?] of x, plus 2m over h squared, e minus v of x, phi of x equals 0. So actually you've proven that phi defined this way solved the same Schrodinger equation with the same energy. So if one is true-- this thing-- I guess we could call it [? three ?] or [? two. ?]

So you've proven that both psi of x and phi of x-- which is equal to psi of minus x are solutions of the Schrodinger equation with the same energy.

And, therefore, if you have two solutions-- and now I emphasize this psi of minus x, and the psi of x. If you have two solutions then you can form the symmetric part of the wave function, which is 1/2 psi of x, plus psi of minus x. And the anti-symmetric part of the wave function, which would be 1/2 of psi of x minus psi of minus x.

And notice that by definition psi s of minus x is indeed psi s of x. It's symmetric. If you change x for minus x on the left hand side, this goes into this, this goes into that. So it's unchanged. Here it's changed by a psi. So psi a of minus x is equal to psi minus psi a of x.

And these two are solutions with the same energy-- psi s and psi a. You see, if you have-- remember that key fact-- if you have two solutions of the Schrodinger equation with the same energy, any linear combination of them is a solution with the same energy.

So we form two linear combinations and they have the same energy. And, therefore, the theorem has been proven. The first part of the theorem-- the wave functions that you work with-- can be chosen to be even or odd. And that's pretty nice.

But now we go to the second part of the statement. So for one-dimensional bound states-- 1-D bound states. Again, there cannot be two solutions. So it cannot be that there are two degenerate solutions. So after all, psi of x and psi of minus x, we have two solutions. This and psi of minus x.

But if you're in a one-dimensional bound state you cannot have two solutions. So they must be proportional to each other. Now, if you started with a solution-- I want to say this. You start with a solution from there, from the beginning you can assume now-- because of property two-- that the solution is real. [? You can work those ?] with real solutions.

So in here, I can assume that psi is real. Just simpler. So these two solutions-- that would be two real solutions-- would be degenerating energy-- there's no degeneracy for bound states. Therefore, these two must be the same up to a constant, that again-- because psi is real c-- is real.

There cannot be two solutions. Let x goes to minus x in this equation. So you would get psi of x equals c of psi of minus x. But psi of minus x uses this equation again. You get c times c psi of x.

But from comparing these two sides, you get that c squared must be equal to 1. But c is real. Therefore, there's only two solutions. Two options. C is equal to plus 1-- in which case-- psi is even-- automatically. Or c is equal to minus 1 and psi is odd.

You have no option. You may think that the general solution of a bound state-- of a symmetric potential-- could be arbitrary. But no. The solutions come out automatically symmetrical or anti-symmetrical.

And that's why-- when we decided to search for all the solutions of the finite square-- well, we could divide it into two cases. Let's find the symmetric solutions and the anti-symmetric solutions. There is no other solution of the Schrodinger equation.

But what if you add a symmetric to an anti-symmetric solution? Don't you get the general solution? Well you cannot add them, because for bound states they are different energies. And adding two solutions with different energies is pointless. It's not a energy eigenstate anymore.

So a very powerful theorem. We'll be using it a lot, and I thought you really ought to see it.