

**PROFESSOR:** Find our final solution, we just have to match the equations. So  $\Psi$  continues at  $x = a$ . And what do we have? Well from two, you have cosine of  $ka$ . And from four you would have  $e^{-ka}$ . This is the value of this-- so the interior solution at  $x = a$  must match the value of the exterior solution of  $k = a$ .  $\Psi'$  must be continuous at  $x = a$  as well.

Well what is the derivative of this function? It's minus the sine of this. So it's  $-k \sin(kx)$ , that becomes  $ka$ , is equal to the derivative of that one, which is  $-\kappa A e^{-\kappa a}$ . Two equations, and how many unknowns? Well there's  $A$  and some information about  $\kappa$  and  $k$ . And the easiest way to eliminate that is to divide them. So you divide the bottom equation by this equation.

So what do we get? Divide the bottom by the top. Minus  $k$  and the minuses cancel, we can cancel those minus signs and you get  $k \tan ka = \kappa A$ . But you already are convinced, I hope, on the idea that we should not use equations that have units. So I will multiply by  $a$  and a little  $a$  to get a [INAUDIBLE] size, and therefore, the right hand side becomes  $\xi = \kappa a$ , and the left side become  $\eta \tan \eta$ .

OK, I want to make a little comment about these quantities already. So all the problem has turned out into the following. You were given a potential and that determines a number  $z_0$ . If you know the width and everything, you know  $z_0$ . Now you have to calculate  $\eta$  and  $\xi$ . If you know either  $\eta$  or  $\xi$ , you know  $\kappa$  or  $k$ . And if you know either  $k$  or  $\kappa$ , since you know  $v_0$ , you will know the energy.

So it's kind of neat to express this more clearly, and I think it's maybe easier if one uses  $\xi$ . And look at  $\xi^2 = \kappa^2 a^2$ . And what is  $\kappa^2$ , it's  $2m |E| / \hbar^2$ . Now you want to find  $E$ , you're going to get in some units. Even  $E$  is nice to have it without units. So I will multiply and divide by  $v_0$ .  $2m v_0 a^2 |E| / \hbar^2$ , absolute value of  $E$  over  $v_0$ .

After all, you probably prefer to know  $E / v_0$ , which tells you how proportional the energy is to the depth of the potential. And this is your famous constant  $z_0$ . So  $E / v_0$  is actually equal to  $\xi^2 / z_0^2$ . And this is something just to keep in mind. If you know  $\xi$ , you certainly must know  $z_0$ , because that's not in your potential, and then you know how much is the energy. All very convenient things.

So punchline for solutions. So what do we have? We have two equations. This equation maybe should be given a number.  $\xi$  equals  $8 \tan \eta$  and  $\eta$  and  $\xi$  squared giving you  $z$  squared. So how do we solve it? We solve it graphically. We have  $\Psi$ ,  $\eta$ , and then we say, oh, let's try to plot the two equations. Well this is a circle.  $\eta$  squared plus  $\Psi$  squared. Now  $\xi$  and  $\eta$  must be positive, so we look at solutions just in this quadrant. Let's put here  $\pi/2$ .  $\pi/3$ ,  $\pi/2$ ,  $2\pi$ , and here is  $\eta$  and there is  $\xi$ .

Well this is a circle, as we said, but let's look at this.  $\xi$  equals  $\eta \tan \eta$ . That vanished as  $\eta$  goes to 0 and will diverge at  $\eta$  equals  $\pi/2$ . So this part, at least, looks like this. And then it will go negative, which don't care, from this region, and then reach here at  $\pi$ . And after  $\pi$ , it will go positive again and it will reach another infinity here. And then at  $3\pi/2$ , at  $2\pi$ , it will go again and reach not another infinity like that.

So these are these curves. And the other curve, the circle, is just a circle here. So, for example, I could have a circle like this. So the radius of this circle is radius  $z_0$ . And there you go, you've solved the problem. At least intuitively you know the answer. And there's a lot of things that come out of this calculation. If the radius  $z_0$  is  $3\pi/2$ , for example, and the radius  $z_0$  represents some potential of some depth and width, there will be just two solutions.

These are these solutions. These points represent values of  $\xi$  and values of  $\eta$ , from which you could read the energy. In fact, you can look at that state and say, that's the state of largest  $\xi$ , and therefore it's the state of the largest absolute value of the energy. It's the most deeply bound state. Then this is next deeply bound state. So there's two bound states in this case. Interestingly, however shallow this potential might be, however small,  $z_0$ , the circle, will always have one intersection, so there will always be at least one solution.

That's the end of that story. Let me say that for the odd case, odd solutions, I will not solve it. It's a good thing to do in recitation or as part of the home work as well. The answer for the odd case is that  $\Psi$  is equal to minus  $\eta$  [ $\cot \eta$ ]  $\eta$ . And in that case, I'll give you a little preview of how the this [ $\cot \eta$ ] looks. It looks like this. And then there are more branches of this thing. So for the odd solution, you have these curves. And if you have a circle, sometimes you don't have a solution. It doesn't intersect this.

So these odd solutions, you will see and try to understand, they don't always exist. You meet a potential that is sufficiently deep to get an odd solution. And then the odds and even solutions will interweave each other and there will be a nice story that you will explore in a lot of detail.

But the is, you've reviewed the problem to unit free calculation, in which you can get the intuition of when solutions exist and when they don't. But solving for the particular numbers are transcendental equations, and you need a computer to solve.