PROFESSOR: Let me do a little exercise using still this manipulation. And I'll confirm the way we think about expectations values.

So, suppose exercise. Suppose you have indeed that psi is equal to alpha i psi i. Compute the expectation value of Q in the state of psi. Precisely, the expectation value of this operator we've been talking about on the state.

So this is equal to the integral dx psi star Q psi. And now I have to put two sums before. And go a little fast here. dx sum over i alpha i psi i star Q sum over j alpha j psi j. No star.

This is equal to sum over i sum over j alpha i star alpha j integral dx psi i star Q psi j. But Q psi j is equal to qj psi j. Therefore, this whole thing is equal to qj times the integral dx of psi i star psi j, which is qj delta ij.

So here we go. It's equal to sum over i, sum over j, alpha i star alpha j, qj delta ij, which is equal to the sum over i. The j's disappear. And this is alpha i squared qi. That's it. OK.

Now you're supposed to look at this and say, yay. Now why is that? Look. How did we define expectation values? We defined it as the sum of the value times the probability that this value have. It's for a random variable.

So here our random variable is the result of the measurement. And what are the possible values? qi's. And what are the probabilities that they have Pi? OK. So the expectation value of q should be that, should be the sum of the possible values times their probabilities, and that's what the system gives.

This is how we defined expectation value of x. Even though it's expectation value of P. And it all comes from the measurement postulate and the definition. Now, this definition and the measurement postulate just shows that this is what we expect. This is the result of the expectation value. OK.

I think I have a nice example. I don't know if I want to go into all the detail of these things, but they illustrate things in a nice way. So let's try to do it.

So here it is. It's a physical example. This is a nice concrete example because things work out. So I think we'll actually illustrate some physical points.

Example. Particle on a circle. x 0 to L. Maybe you haven't seen a circle described by that, but you take the x-axis, and you say yes, the circle is 0 to L. L and 0. And the way you think of it is that this point is identified with this point.

If you have a line and you identify the two endpoints, that's called a circle. It's in the sense of topology. A circle as the set of points equidistant to a center is a geometric description of a round circle. But this, topologically speaking, anything that is closed is topologically a circle. We think of a circle as this, physically, or it could be a curved line that makes it into a circle. But it's not important.

Let's consider a free particle on a circle, and suppose the circle has an end L. So x belongs here. And here is the wave function, psi equals 2 over L, 1 over square root of 3 sine of 2 pi x over L, plus 2 over square root of 3 cosine 6 pi x over L. This is the wave function of your particle on a circle.

At some time, time equals 0, it's a free particle. No potential. And it lives in the circle, and these functions are kind of interesting. You see, if you live on the circle you would want to emphasize the fact that this point 0 is the same as the point L, so you should have that psi and L must be equal to psi at 0. It's a circle, after all, it's the same point.

And therefore for 0 or for L, the difference here is 0 or 2 pi, and the sine is the same thing. And 0, when x equals 0, and 6 pi, so that's also periodic, and it's fine. It's a good wave function result.

The question is, for this problem, what are, if you measure momentum, measure momentum, what are the possible values and their probabilities? Probabilities. So you decide to measure momentum of this particle. What can you get? OK.

It looks a little nontrivial, and it is a little nontrivial. Momentum. So I must sort of find the momentum eigenstates. Momentum eigenstates, they are those infinite plane waves, e to the ikx, that we could never normalize. Because you square it, it's 1, and the integral over all space is infinite. So are we heading for disaster here? No. Because it lives in a finite space.

Yes, you have a question?

STUDENT: Should it be a wave function [INAUDIBLE] complex? Because right now, it just looks like it's a real value. And we can't [INAUDIBLE] real wave functions, can we?

PROFESSOR: Well, it is the wave function at time equals 0. So the time derivative would have to bring in complex things. So you can have a wave function that is 0, that is real at some particular time. Like, any wave function psi of x e to the minus iEt over h bar is a typical wave function. And then at time equal 0 it may be real. It cannot be real forever. So you cannot assume it's real. But at some particular times it could be real. Very good question.

The other thing you might say, look, this is too real to have momentum. Momentum has to do with waves. That's probably not a reliable argument.

OK, so, where do we go from here? Well, let's try to find the momentum eigenstates. They should be things like that, exponentials. So how could they look? Well, e to the 2 pi i, maybe. What else? x, there should be an x for a momentum thing. Now there should be no units here, so there better be an L here. And now I could put, maybe, well the 2 maybe was-- why did I think of the 2 or the pi? Well, for convenience. But let's see what.

Suppose you have a number m here. Then the good thing about this is that when x is equal to 0, there is some number here, but when x is equal to L, it's a multiple of e to the 2 pi i, so that's periodic. So this does satisfy, I claim, it's the only way if m is any integer. So it goes from minus infinity to infinity. Those things are periodic. They satisfy psi. Actually they satisfy psi of x plus L is equal to psi of x.

OK. That seems to be something that could be a momentum eigenstate. And then I have to normalize it. Well, if I square it and integrate it. If I square it then the phase cancels, so you get 1. If you integrate it you get L. If you put 1 over the square root of L, when you square it and integrate, you will get 1. So here it is. Psi m's of x are going to be defined to be this thing. And I claim these things are momentum eigenstates.

In fact, what is the value of the momentum? Well, you calculate h bar over i d dx on psi m. And you get what? You get 2 pi m over L times h bar times psi. The h bar is there, the i cancels, and everything then multiplies, the x falls down. So this is the state with momentum P equals to h bar 2 pi m over L.

OK. Actually, doing that, we've done the most difficult part of the problem. You've found the momentum eigenfunctions. So now the rest of the thing is to rewrite this in terms of this kind of objects. I'll do it in a second. Maybe I'll leave a little space there and you can check the algebra, and you can see it in the notes. But you know what you're supposed to do.

A sine of x is e to the ix minus is e to the minus ix over 2i. So you'd get these things converted to exponentials. The cosine of x is equal to e to the ix plus e to the minus ix over 2. So if you do that with those things, look. What the sine of 2 pi x going to give you? It's going to give you some exponentials of 2 pi ix over L.

So suppose that m equals 1. And m Equals minus 1. And this will give you m equals 3, 3 times 2 is 6. And m equal minus 3.

So I claim, after some work, and you could try to do it. I think it would be a nice exercise. Psi is equal square root of 2 over 3, 1 over 2 i psi 1 minus square root of 2 over 3, 1 over 2i psi minus 1 plus 1 over square root of 3 psi 3, plus 1 over square root of 3 psi minus 3. And it should give you some satisfaction to see something like that. You're now seeing the wave function written as a superposition of momentum eigenstates. This theorem came through.

In this case, as a particle in the circle, the statement is that the eigenfunctions are the exponentials, and it's Fourier's theorem. Again, for a series.

So finally, here is the answer. So psi 1, we can measure psi 1. What is the momentum of psi 1? So here are p values. And probabilities. The first value, psi 1, the momentum is 2 pi h bar over L. So 2 pi h bar over L. And what is its probability? It's this whole number squared. So square root of 2/3, 1 over 2i squared. So how much is that? It's 2/3 times 1/4. 2/3 times 1/4, which is 1/6.

And the other value that you can get is minus this one, so minus 2 pi h bar over L. This minus doesn't matter, probability also 1/6. The next one is with 3. So you can get 2, 6 pi, 6 pi h bar over L, with probability square of this, 1/3. And minus 6 pi h bar over L with probability 1/3. Happily our probabilities add up.

So there you go. That's the theorem expressed in a very clear example. We had a wave function. You wrote it as a sum of four momentum eigenstates. And now you know, if you do a measurement, what are the possible values of the momentum. This should have been probably 1/6. You can do anything you want.