Lecture 4

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1 de Broglie wavelength and Galilean transformations

We have seen that to any free particle with momentum $\mathbf{p}$, we can associate a plane wave, or a "matter wave", with de Broglie wavelength $\lambda = h/p$, with $p = |\mathbf{p}|$. The question is, *waves of what*? Well, this wave is eventually recognized as an example of what one calls the *wavefunction*. The wavefunction, as we will see is governed by the Schrödinger equation. As we have hinted, the wavefunction gives us information about probabilities, and we will develop this idea in detail.

Does the wave have directional or polarization properties like electric and magnetic fields in an electromagnetic wave? Yes, there is an analog of this, although we will not delve into it now. The analog of polarization corresponds to spin! The effects of spin are negligible in many cases (small velocities, no magnetic fields, for example) and for this reason, we just use a scalar wave, a complex number

$$\Psi(\mathbf{x}, t) \in \mathbb{C} \tag{1.1}$$

that depends on space and time. A couple of obvious questions come to mind. Is the wavefunction measurable? What kind of object is it? What does it describe? In trying to get intuition about this, let's consider how different observers perceive the de Broglie wavelength of a particle, which should help us understand what kind of waves we are talking about. Recall that

$$p = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k, \tag{1.2}$$

where k is the wavenumber. How would this wave behave under a change of frame?

We therefore consider two frames S and S' with the x and x' axes aligned and with S' moving to the right along the $+x$ direction of S with constant velocity v. At time equal zero, the origins of the two reference frames coincide.

The time and spatial coordinates of the two frames are related by a *Galilean transformation*, which states that

$$x' = x - vt, \quad t' = t\,. \tag{1.3}$$

Indeed time runs at the same speed in all Galilean frames and the relation between x and x' is manifest from the arrangement shown in Fig. 1.

Now assume both observers focus on a particle of mass m moving with nonrelativistic speed. Call the speed and momentum in the S frame $\widetilde{v}$ and $p = m\widetilde{v}$, respectively. It follows by differentiation with

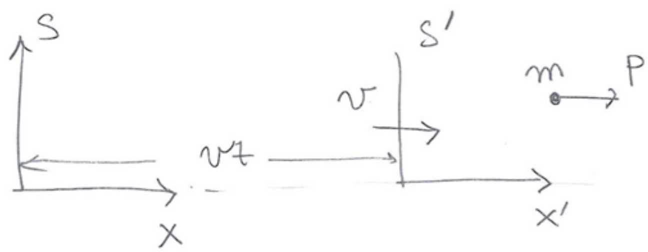

Figure 1: The S' frame moves at speed v along the x-direction of the S frame. A particle of mass m moves with speed $\widetilde{v}$, and thus momentum $p = m\widetilde{v}$, in the S frame.

respect to $t = t'$ of the first equation in (1.3) that

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v\,, \tag{1.4}$$

which means that the particle velocity $\widetilde{v}'$ in the S' frame is given by

$$\widetilde{v}' = \widetilde{v} - v\,. \tag{1.5}$$

Multiplying by the mass m we find the relation between the momenta in the two frames

$$p' = p - mv. \tag{1.6}$$

The momentum p' in the S' frame can be appreciably different from the momentum p in the S frame. Thus the observers in S' and in S will obtain rather different de Broglie wavelengths λ' and λ! Indeed,

$$\lambda' = \frac{h}{p'} = \frac{h}{p - mv} \neq \lambda, \tag{1.7}$$

This is very strange! As we review now, for ordinary waves that propagate in the rest frame of a medium (like sound waves or water waves) Galilean observers will find frequency changes but no change in wavelength. This is intuitively clear: to find the wavelength one need only take a picture of the wave at some given time, and both observers looking at the picture will agree on the value of the wavelength. On the other hand to measure frequency, each observers must wait some time to see a full period of the wave go through them. This will take different time for the different observers.

Let us demonstrate these claims quantitatively. We begin with the statement that the phase $\phi = kx - \omega t$ of such a wave is a Galilean invariant. The wave itself may be $\cos\phi$ or $\sin\phi$ or some combination, but the fact is that the physical value of the wave at any point and time must be agreed by the two observers. The wave is an observable. Since all the features of the wave (peaks, zeroes, etc, etc) are controlled by the phase, the two observers must agree on the value of the phase.

In the S frame the phase can be written as follows

$$\phi = kx - \omega t = k(x - \tfrac{\omega}{k}t) = \frac{2\pi}{\lambda}(x - Vt) = \frac{2\pi x}{\lambda} - \frac{2\pi V}{\lambda}t, \tag{1.8}$$

where $V = \frac{\omega}{k}$ is the wave velocity. Note that the wavelength is read from the coefficient of x and ω is minus the coefficient of t The two observers should agree on the value of ϕ. That is, we should have

$$\phi'(x', t') = \phi(x, t) \tag{1.9}$$

where the coordinates and times are related by a Galilean transformation. Therefore

$$\phi'(x',t') \;=\; \frac{2\pi}{\lambda}(x - Vt) \;=\; \frac{2\pi}{\lambda}(x' + vt' - Vt') = \frac{2\pi}{\lambda}x' - \frac{2\pi(V-v)}{\lambda}t'. \tag{1.10}$$

Since the right-hand side is expressed in terms of primed variables, we can read λ' from the coefficient of x' and ω' as minus the coefficient of t':

$$\lambda' = \lambda \tag{1.11}$$

$$\omega' = \frac{2\pi}{\lambda}(V-v) = \frac{2\pi V}{\lambda}\left(1 - \frac{v}{V}\right) = \omega\left(1 - \frac{v}{V}\right). \tag{1.12}$$

This confirms that, as we claimed, for a physical wave propagating in a medium, the wavelength is a Galilean invariant and the frequency transforms.

So what does it mean that the wavelength of matter waves change under a Galilean transformation? It means that the Ψ waves are not directly measurable! Their value does not correspond to a measurable quantity for which all Galilean observers must agree. Thus, the wavefunction need not be invariant under Galilean transformations:

$$\Psi(x,t) \;\neq\; \Psi'(x',t'), \tag{1.13}$$

where (x,t) and (x',t') are related by Galilean transformations and thus represent the same point and time. You will figure out in Homework the correct relation between $\Psi(x,t)$ and $\Psi'(x',t')$.

What is the frequency ω of the de Broglie wave for a particle with momentum p? We had

$$p = \hbar k \tag{1.14}$$

which fixes the wavelength in terms of the momentum. The frequency ω of the wave is determined by the relation

$$E \;=\; \hbar\omega\,, \tag{1.15}$$

which was also postulated by de Broglie and fixes ω in terms of the energy E of the particle. Note that for our focus on non-relativistic particles the energy E is determined by the momentum through the relation

$$E \;=\; \frac{p^2}{2m}\,. \tag{1.16}$$

We can give three pieces of evidence that (1.15) is a reasonable relation.

1. If we superpose matter waves to form a wave-packet that represents the particle, the packet will move with the so called group velocity v_g, which in fact coincides with the velocity of the particle. The group velocity is found by differentiation of ω with respect to k, as we will review soon:

$$v_g \;=\; \frac{d\omega}{dk} \;=\; \frac{dE}{dp} \;=\; \frac{d}{dp}\Big(\frac{p^2}{2m}\Big) \;=\; \frac{p}{m} \;=\; v\,. \tag{1.17}$$

2. The relation is also suggested by special relativity. The energy and the momentum components of a particle form a four-vector:

$$\left(\frac{E}{c}\,,\,p\right) \tag{1.18}$$

Similarly, for waves whose phases are relativistic invariant we have another four-vector

$$\left(\frac{\omega}{c}\,,\,k\right) \tag{1.19}$$

Setting two four-vectors equal to each other is a consistent choice: it would be valid in all Lorentz frames. As you can see, both de Broglie relations follow from

$$\left(\frac{E}{c}\,,\,p\right) \;=\; \hbar\,\left(\frac{\omega}{c}\,,\,k\right)\,. \tag{1.20}$$

3. For photons, (1.15) is consistent with Einstein's quanta of energy, because $E = h\nu \;=\; \hbar\omega$.

In summary we have

$$\boxed{p = \hbar k, \quad E = \hbar\omega\,.} \tag{1.21}$$

These are called the *de Broglie relations*, and they are valid for all particles.

2 Phase and Group Velocities

To understand group velocity we form wave packets and investigate how fast they move. For this we will simply assume that $\omega(k)$ is some arbitrary function of k. Consider a superposition of plane waves $e^{i(kx-\omega(k)t)}$ given by

$$\psi(x,t) = \int dk\; \Phi(k) e^{i(kx-\omega(k)t)}. \tag{2.22}$$

We assume that the function $\Phi(k)$ is peaked around some wavenumber $k = k_0$, as shown in Fig. 2.

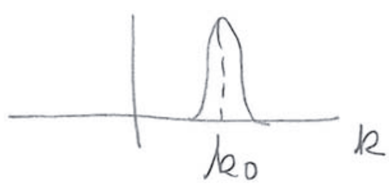

Figure 2: The function $\Phi(k)$ is assumed to peak around $k = k_0$.

In order to motivate the following discussion consider the case when $\Phi(k)$ not only peaks around k_0 but it also is *real* (we will drop this assumption later). In this case the phase φ of the integrand comes only from the exponential:

$$\varphi(k) = \; kx - \omega(k)t\,. \tag{2.23}$$

We wish to understand what are the values of x and t for which the packet $\psi(x,t)$ takes large values. We use the *stationary phase principle*: since only for $k \sim k_0$ the integral over k has a chance to give a non-zero contribution, the phase factor must be *stationary* at $k = k_0$. The idea is simple: if a function is multiplied by a rapidly varying phase, the integral washes out. Thus the phase must have zero derivative at k_0. Applying this idea to our phase we find the derivative and set it equal to zero at k_0:

$$\frac{d\varphi}{dk}\Big|_{k_0} \;=\; x - \frac{d\omega}{dk}\Big|_{k_0} t \;=\; 0\,. \tag{2.24}$$

This means that $\psi(x,t)$ is appreciable when x and t are related by

$$x = \frac{d\omega}{dk}\Big|_{k_0} t \,, \tag{2.25}$$

showing that the packet moves with *group velocity*

$$v_g = \frac{d\omega}{dk}\Big|_{k_0} . \tag{2.26}$$

Exercise. If $\Phi(k_0)$ is not real write $\Phi(k) = |\Phi(k)|e^{i\phi(k)}$. Find the new version of (2.25) and show that the velocity of the wave is not changed.

Let us now do a more detailed calculation that confirms the above analysis and gives some extra insight. Notice first that

$$\psi(x,0) = \int dk \; \Phi(k)e^{ikx} \,. \tag{2.27}$$

We expand $\omega(k)$ in a Taylor expansion around $k = k_0$

$$\omega(k) = \omega(k_0) + (k-k_0)\,\frac{d\omega}{dk}\Big|_{k_0} + \mathcal{O}\left((k-k_0)^2\right) . \tag{2.28}$$

Then we find, neglecting the $\mathcal{O}((k-k_0)^2)$ terms

$$\psi(x,t) = \int dk \; \Phi(k)\, e^{ikx}\, e^{-i\omega(k_0)t} e^{-i(k-k_0)\,\frac{d\omega}{dk}\big|_{k_0} t} . \tag{2.29}$$

It is convenient to take out of the integral all the factors that do not depend on k:

$$\begin{aligned} \psi(x,t) &= e^{-i\omega(k_0)t + ik_0\,\frac{d\omega}{dk}\big|_{k_0} t} \int dk \; \Phi(k) e^{ikx} e^{-ik\,\frac{d\omega}{dk}\big|_{k_0} t} \\ &= e^{-i\omega(k_0)t + ik_0\,\frac{d\omega}{dk}\big|_{k_0} t} \int dk \; \Phi(k) e^{ik\left(x - \frac{d\omega}{dk}\big|_{k_0} t\right)} . \end{aligned} \tag{2.30}$$

Comparing with (2.27) we realize that the integral in the above expression can be written in terms of the wavefunction at zero time:

$$\psi(x,t) = e^{-i\omega(k_0)t + ik_0\,\frac{d\omega}{dk}\big|_{k_0} t} \;\; \psi\Big(x - \frac{d\omega}{dk}\Big|_{k_0} t\Big). \tag{2.31}$$

The phase factors in front of the expression are not important in tracking where the wave packet is. In particular we can take the norm of both sides of the equation to find

$$|\psi(x,t)| = \Big|\psi\Big(x - \frac{d\omega}{dk}\Big|_{k_0} t, 0\Big)\Big|. \tag{2.32}$$

If $\psi(x,0)$ peaks at some value x_0 it is clear from the above equation that $|\psi(x,t)|$ peaks for

$$x - \frac{d\omega}{dk}\Big|_{k_0} t = x_0 \quad \rightarrow \quad x = x_0 + \frac{d\omega}{dk}\Big|_{k_0} t \,, \tag{2.33}$$

showing that the peak of the packet moves with velocity $v_{\rm gr} = \frac{d\omega}{dk}$, evaluated at k_0.

3 Choosing the wavefunction for a free particle

What is the mathematical form of the wave associated with a particle a particle with energy E and momentum p? We know that ω and k are determined from $E = \hbar omega$ and $p = \hbar k$. Let's suppose that we want our wave to be propagating in the $+\hat{x}$ direction. All the following are examples of waves that could be candidates for the particle wavefunction.

1. $\sin(kx - \omega t)$
2. $\cos(kx - \omega t)$
3. $e^{i(kx-\omega t)} = e^{ikx}e^{-i\omega t}$ - time dependence $\propto e^{-i\omega t}$
4. $e^{-i(kx-\omega t)} = e^{-ikx}e^{i\omega t}$ - time dependence $\propto e^{+i\omega t}$

In the third and fourth options we have indicated that the time dependence could come with either sign. We will use superposition to decide which is the right one! We are looking for a wave-function which is non-zero for all values of x.

Let's take them one by one:

1. Starting from (1), we build a superposition in which the particle has equal probability to be found moving in the $+x$ and the $-x$ directions.

   $$\Psi(x,t) = \sin(kx - \omega t) + \sin(kx + \omega t) \tag{3.1}$$

   Expanding the trigonometric functions this can be simplified to

   $$\Psi(x,t) \;=\; 2\sin(kx)\cos(\omega t)\,. \tag{3.2}$$

   But this result is not sensible. The wave function vanishes identically for all x at some special times

   $$\omega t = \left(\tfrac{\pi}{2}, \tfrac{3\pi}{2}, \tfrac{5\pi}{2}, ...\right) \tag{3.3}$$

   A wavefunction that is zero cannot represent a particle.

2. Constructing a wave function from (2) with a superposition of left and right going cos waves,

   $$\Psi(x,t) = \cos(kx - \omega t) + \cos(kx + \omega t) = 2\cos(kx)\cos(\omega t)\,. \tag{3.4}$$

   This choice is no good, it also vanishes identically when $\omega t = \left(\frac{\pi}{2}, \frac{3\pi}{2}, ...\right)$

3. Let's try a similar superposition of exponentials from (3), with both having the same time dependence

   $$\Psi(x,t) \;=\; e^{i(kx-\omega t)} + e^{i(-kx-\omega t)} \tag{3.5}$$
   $$\;=\; \left(e^{ikx} + e^{-ikx}\right)e^{-i\omega t} \tag{3.6}$$
   $$\;=\; 2\cos kx\, e^{-i\omega t}\,. \tag{3.7}$$

   This wavefunction meets our criteria! It is never zero for all values of x because $e^{-i\omega t}$ is never zero.

4. A superposition of exponentials from (4) also meets our criteria

$$\begin{aligned}\Psi(x,t) &= e^{-i(kx-\omega t)} + e^{-i(-kx-\omega t)} && (3.8)\\ &= \left(e^{ikx} + e^{-ikx}\right) e^{i\omega t} && (3.9)\\ &= 2\cos kx\, e^{i\omega t}\,. && (3.10)\end{aligned}$$

This is never zero for all values of x

Since both options (3) and (4) seem to work we ask: Can we use *both* (3) and (4) to represent a particle moving to the right (in the $+\hat{x}$ direction)? Let's assume that we can. Then, since adding a state to itself should not change the state, we could represent the right moving particle by using the sum of (3) and (4)

$$\Psi(x,t) = e^{i(kx-\omega t)} + e^{-i(kx-\omega t)} = 2\cos(kx-\omega t)\,. \tag{3.11}$$

This, however, is the same as (2), which we already showed leads to difficulties. Therefore we must choose between (3) and (4).

The choice is a matter of convention, and all physicists use the same convention. We take the free particle wavefunction to be

$$\boxed{\text{Free particle wavefunction :} \quad \Psi(x,t) \;=\; e^{i(kx-\omega t)}\,,} \tag{3.12}$$

representing a particle with

$$p = \hbar k\,, \quad \text{and} \quad E = \hbar\omega\,. \tag{3.13}$$

In three dimensions the corresponding wavefunction would be

$$\boxed{\text{Free particle wavefunction :} \quad \Psi(\mathbf{x},t) \;=\; e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,,} \tag{3.14}$$

representing a particle with

$$\mathbf{p} = \hbar\,\mathbf{k}\,, \quad \text{and} \quad E = \hbar\omega\,. \tag{3.15}$$

Andrew Turner and Sarah Geller transcribed Zwiebach's handwritten notes to create the first LaTeX version of this document.