8.04, Quantum Physics I, Fall 2015

TEST 2

Tuesday November 17, 9:30-11:00 pm

You have 90 minutes.

Answer all problems in the white books provided. Write YOUR NAME and YOUR SECTION on your white book(s).

There are five questions, totalling 100 points.

No books, notes, or calculators allowed.

Show your work CLEARLY!

Formula Sheet

- $\hbar c \simeq 197.3 \text{ MeV} \cdot \text{fm}\,, \quad m_e c^2 \simeq 0.511 \text{ MeV}\,, \quad m_p c^2 = 938 \text{ MeV}\,, \quad \frac{e^2}{\hbar c} \simeq \frac{1}{137}$
- Relativity: $p = \gamma\, mv\,, \quad E = \gamma mc^2\,, \quad E^2 = p^2c^2 + m^2c^4\,, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}\,, \quad \beta = \frac{v}{c}$
- Photons: $E = h\nu\,, \; p = \frac{h}{\lambda}\,,$ or $E = \hbar\omega\,, \; p = \hbar k$
- Wavelengths

$$\text{de Broglie: } \lambda = \frac{h}{p}\,, \qquad \text{Compton: } \lambda_C = \frac{h}{mc}\,.$$

- Momentum and position operators

$$p \;=\; \frac{\hbar}{i}\frac{\partial}{\partial x}\,, \quad [x, p\,] \;=\; i\hbar\,, \qquad \mathbf{p} \;=\; \frac{\hbar}{i}\nabla\,, \quad [x_i, p_j\,] = i\hbar\,\delta_{ij}$$

- Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}(\mathbf{x},t) \;=\; \Big(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x},t)\Big)\Psi(\mathbf{x},t)\,,$$

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t) + \nabla\cdot\mathbf{J}(\mathbf{x},t) = 0$$

$$\rho(\mathbf{x},t) = |\Psi(\mathbf{x},t)|^2\;; \qquad \mathbf{J}(\mathbf{x},t) = \frac{\hbar}{m}\text{Im}\,[\Psi^*\nabla\Psi]$$

- Fourier transforms:

$$\Psi(x) = \frac{1}{\sqrt{2\pi}}\int dk\,\Phi(k)e^{ikx}\,, \quad \Phi(k) = \frac{1}{\sqrt{2\pi}}\int dx\,\Psi(x)e^{-ikx}\,, \quad \int dx\,|\Psi(x)|^2 = \int dk\,|\Phi(k)|^2$$

$$\Psi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\,\Phi(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,, \;\; \Phi(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3x\,\Psi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \;\; \int d^3x\,|\Psi(\mathbf{x})|^2 = \int d^3k\,|\Phi(\mathbf{k})|^2$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dx = \delta(k)\,, \qquad \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3x = \delta^{(3)}(\mathbf{k})$$

$$\int_{-\infty}^{+\infty} dx\,\exp\left(-ax^2 + bx\right) = \sqrt{\frac{\pi}{a}}\,\exp\Big(\frac{b^2}{4a}\Big)\,, \quad \text{when } \text{Re}(a) > 0\,.$$

- Wavepackets

$$v_{group} \;=\; \frac{d\omega}{dk}\,, \quad \Delta k\,\Delta x \simeq 1\,, \quad \text{shape preserving}: \; t\,\Delta v \leq \Delta x$$

- Hermitian conjugation:

$$\int dx\,(K\Psi(x,t))^*\Psi(x,t) \;=\; \int dx\,\Psi^*(x,t)\,(K^\dagger\Psi(x,t))$$

If $K^\dagger = K$, then K is Hermitian.

- Expectation values

$$\langle Q\rangle(t) \;=\; \int dx\,\Psi^*(x,t)(Q\Psi(x,t))$$

- Time evolution of expectation value. For Q Hermitian

$$i\hbar\frac{d}{dt}\langle Q\rangle \;=\; \big\langle [Q,H]\big\rangle$$

- Commutator identity

$$[A,BC]=[A,B]C+B[A,C]$$

- Uncertainty ΔQ of a Hermitian operator Q

$$(\Delta Q)^2 \;=\; \langle Q^2\rangle-\langle Q\rangle^2 \;=\; \big\langle (Q-\langle Q\rangle)^2\big\rangle$$

- Uncertainty principle: $\Delta x\,\Delta p \geq \frac{\hbar}{2}$

- Stationary state:

$$\Psi(\mathbf{x},t)=\psi(\mathbf{x})e^{-iEt/\hbar}\,, \qquad -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x})+V(\mathbf{x})\psi(\mathbf{x})=E\,\psi(\mathbf{x})$$

- Infinite square well

$$V(x)=\begin{cases}0\,, & \text{for } \; 0<x<a,\\ \infty & \quad\text{otherwise}\end{cases}$$

$$\psi_n(x)=\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\,, \qquad E_n=\frac{\hbar^2\pi^2n^2}{2ma^2}\,, \qquad n=1,2,\ldots$$

- Finite square well bound states: $E\leq 0$

$$V(x)=\begin{cases}-V_0\,, & \text{for } \; |x|<a, \quad V_0>0\\ \;\;0 & \text{for } \; |x|>a\end{cases}$$

$$\eta^2 \;\equiv\; \frac{2m(E+V_0)a^2}{\hbar^2}\,, \quad \xi^2 \;\equiv\; \frac{2m|E|a^2}{\hbar^2}\,, \quad z_0^2 \;\equiv\; \frac{2mV_0a^2}{\hbar^2}$$

$$\rightarrow \quad \frac{|E|}{V_0} \;=\; \frac{\xi^2}{z_0^2}\,, \qquad \xi^2+\eta^2 \;=\; z_0^2$$

$$\text{Even solutions:} \quad \xi \;=\; \eta\tan\eta$$

$$\text{Odd solutions:} \quad \xi \;=\; -\eta\cot\eta$$

- Delta function potential:

$$V=-\alpha\,\delta(x), \quad \alpha>0, \qquad \text{Bound state:} \quad E=-\frac{m\alpha^2}{2\hbar^2}$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\,(\hat{N} + \tfrac{1}{2})\,, \quad \hat{N} = \hat{a}^\dagger\hat{a}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\ , \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\ ,$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)\ , \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})\,,$$

$$[\hat{x}, \hat{p}] = i\hbar\,, \quad [\hat{a}, \hat{a}^\dagger] = 1\ , \qquad [\hat{N}, \hat{a}\,] = -\hat{a}\,, \quad [\hat{N}\,, \hat{a}^\dagger\,]\ =\ \hat{a}^\dagger\,.$$

$$\hat{a}\phi_0\ =\ 0\,, \quad \phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\ .$$

$$\phi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n\phi_0$$

$$\hat{H}\,\phi_n = E_n\,\phi_n = \hbar\omega\big(n + \tfrac{1}{2}\big)\,\phi_n\,, \quad \hat{N}\,\phi_n = n\,\phi_n\,, \quad (\phi_m\,, \phi_n)\ =\ \delta_{mn}$$

$$\hat{a}^\dagger\phi_n = \sqrt{n+1}\,\phi_{n+1}\ , \quad \hat{a}\,\phi_n = \sqrt{n}\,\phi_{n-1}\ .$$

1. **Sketching wavefunctions** [20 points]

   A symmetric potential is attached to the first inside page of your blue book (it is also the last page on this test.) The potential is infinite for $|x| > a$ and it is an even function of x. Indicated on the figure as horizontal dashed lines are the first (ground state), the second, and the fifth energy levels. Sketch the associated wavefunctions. Pay attention to symmetry, convexity or concavity, inflection points, nodes, amplitude, and wavelength.

2. **Cleaning up units** [15 points]

   In the hydrogen atom, the length scale is the Bohr radius a_0 given by

   $$a_0 = \frac{\hbar^2}{me^2}\,.$$

   Consider now the "radial equation" for the radial part $\psi(r)$ of the wavefunction:

   $$\Big(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2\ell(\ell+1)}{2mr^2} - \frac{e^2}{r}\Big)\psi(r) \;=\; E\psi(r)\,.$$

   Here ℓ is a non-negative integer. Clean up the equation by defining a unit-free coordinate u and a unit-free energy $\mathcal{E}$ so that the equation will take the form:

   $$\Big(-\frac{d^2}{du^2} \;+\; \ldots\;\Big)\psi(u) \;=\; \mathcal{E}\,\psi(u)\,.$$

   (a) How are r and u related?

   (b) How are $\mathcal{E}$ and E related?

   (c) Complete the above equation.

3. **Exercises with the harmonic oscillator** [15 points]

   (a) Calculate $\big(\phi_n, (\hat{x})^3\phi_n\big)$.

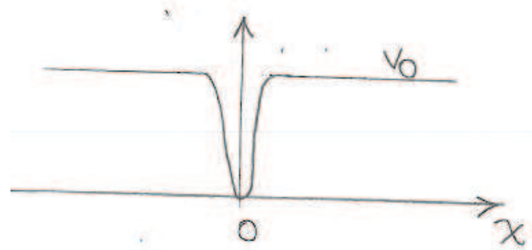
   (b) Calculate $\big(\phi_0, (\hat{x})^{10}\phi_{10}\big)$.

4. **Ground state and first excited state of a potential** [20 points]

Consider the even potential sketched below, with height V_0 except for a dip around $x = 0$, near which the potential is accurately described by a quadratic function:

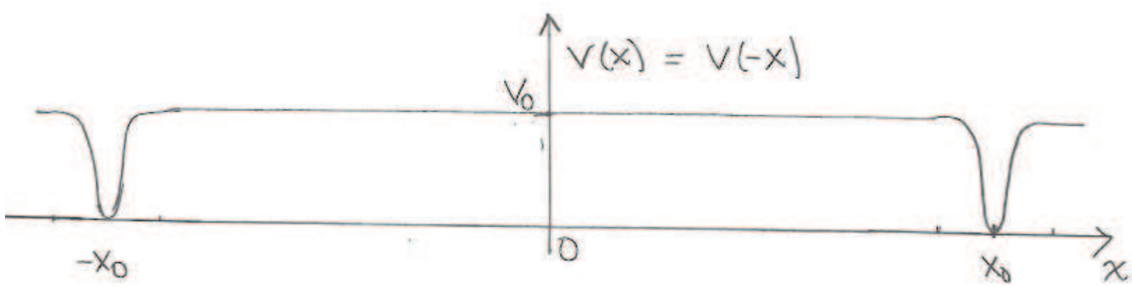
$$V(x) \simeq \tfrac{1}{2}\alpha x^2 , \quad x \text{ near } 0 ,$$

where $\alpha > 0$ is a constant.



(a) Use the harmonic oscillator to give an estimate for the ground state energy. Find an inequality satisfied by α, V_0, m, and $\hbar$ required for the accuracy of your result.

(b) Now consider the even potential built by using two copies of the potential above, with well-separated centers at $\pm x_0$.



Give the approximate energies of the ground state and the first excited state of this potential and sketch the associated wavefunctions. Write an inequality involving $x_0, \alpha, \hbar, m$ required for the accuracy of your result.

(c) Combine your two inequalities into the form

$$\ldots \ll \alpha \ll \ldots$$

where the dots represent quantities you must write out.

5. **From square well to delta function** [30 points]

In this problem we ask you to derive the bound state energy of a particle of mass m on the delta function potential

$$V_\delta(x) = -\alpha\,\delta(x)\,, \quad \alpha > 0\,,$$

starting from the problem of a particle of mass m on a finite square well potential. For this purpose consider a square well with potential $V_a(x)$:

$$V_a(x) = \begin{cases} -V_0\,, & \text{for } |x| < a, \quad V_0 > 0 \\ 0 & \text{for } |x| > a \end{cases}$$

We think of the total width $2a$ as a *regulator*, namely, a parameter that will be taken to zero in the limit as the potential V_a becomes infinitesimally narrow to represent the delta function potential V_δ.

You can think of a negative delta function as the limit of a well whose width and height are simultaneously going to zero and infinity, respectively, while keeping the area of the well equal to one.

If the regulator works properly, the final answer for the energy of the delta function must not depend on a.

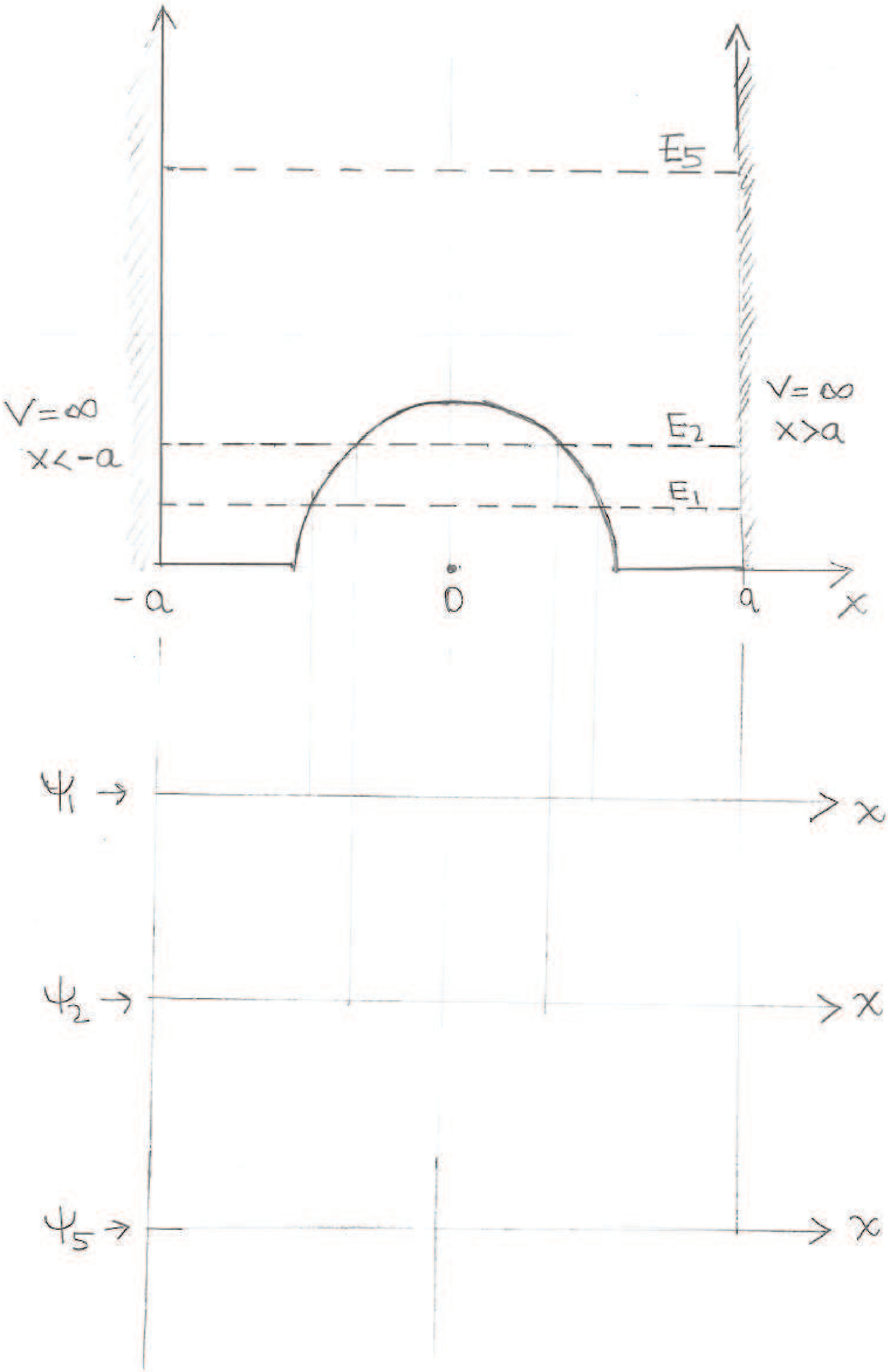
(a) For a given value of a fix the value of V_0 so that in the limit $a \to 0$ the potential V_a represents V_δ correctly. Give your answer in terms of α and a.

(b) What is the value of z_0^2 for the well V_a? As $a \to 0$ what happens to z_0? Explain why such behavior is reasonable.

(c) Work with a very small but nonzero, and calculate *leading* approximations for η and ξ in terms of z_0.

(d) Determine the bound state energy of the delta function potential from your analysis of the well. Do you get the correct answer?

NOTE: Being a proof-like problem, show clear work!


E5
V=∞
x<-a
E2
E1
V=∞
x>a
-a
0
a
x
ψ1 →
x
ψ2 →
x
ψ5 →
x