Quantum Physics I (8.04) Spring 2016
Assignment 10

MIT Physics Department — *Due Friday May 6, 2016*
29 April, 2016 — *12:00 noon*

Reading: Griffiths sections 4.1, 4.2, and 4.3.

Problem Set 10

1. **Bound states from imaginary wavenumber** [5 points]

   Consider the scattering solution for a finite range one-dimensional potential:

   $$\psi_x(x) = e^{i\delta(k)} \sin(kx + \delta(k)) \,, \quad x > R \,.$$

   Show that having a bound state means $A_s = e^{i\delta} \sin\delta$ has a pole at $k = i\kappa$ with $\kappa > 0$.

2. **Simultaneous eigenfunctions** [5 points]

   Consider two Hermitian operators $\hat{A}$ and $\hat{B}$ that commute. Assume that at least one of the operators, say $\hat{A}$, has no degeneracies in its spectrum. Show that the eigenfunctions of $\hat{A}$ are also eigenfunctions of $\hat{B}$.

3. **Expectation values in a particular wavefunction** [10 points]. (Based on Ohanian, Ch. 7, problem 17).

   Suppose a particle has the wavefunction

   $$\psi(r, \theta, \phi) \;=\; \frac{1}{4}\sqrt{\frac{5}{\pi}}\, \sin^2\theta \big(1 + \sqrt{14}\cos\theta\big) \cos 2\phi \; f(r) \,,$$

   with $f(r)$ a normalized radial wavefunction.

   (a) Rewrite this wavefunction in terms of spherical harmonics. What are the possible outcomes of the measurement of L^2 and L_z? What are the corresponding probabilities?

   (b) What are the expectation values of L^2 and L_z?

   (c) Determine the uncertainties ΔL^2 and ΔL_z.

4. **Spherical wells.** [10 points]

(a) Consider $\ell = 0$ states of a particle moving in the *infinite* spherical well

$$V(r) \;=\; \begin{cases} 0\,, & \text{if } r < a \\ \infty\,, & \text{if } r > a\,. \end{cases}$$

Solve the radial equation for the radial wavefunction $u(r)$ and find the possible energy levels. Recall that u must vanish at $r = 0$. Try matching this $\ell = 0$ spectrum to that of a one-dimensional infinite well potential $V(x)$.

(b) Consider now states of a particle moving in a *finite* spherical well with $V_0 > 0$:

$$V(r) \;=\; \begin{cases} -V_0 & \text{if } r < a \\ 0\,, & \text{if } r > a\,. \end{cases}$$

Show that there is no bound state if

$$V_0 a^2 \;<\; \frac{\pi^2\hbar^2}{8m}\,.$$

5. **Hydrogen atom with total momentum.** [10 points] Based on Ohanian.

When the motion of the nucleus is taken into account, the state of the hydrogen atom can be represented by a wavefunction $\psi(\mathbf{X}, \mathbf{x})$, with $\mathbf{X}$ the center of mass coordinate and $\mathbf{x} = \mathbf{x}_e - \mathbf{x}_p$ the relative coordinate pointing from the proton to the electron.

Suppose that the atom is in such a state that the *total* momentum has equal probabilities for the values $\mathbf{p}_0$ and $-\mathbf{p}_0$. Moreover the internal states are $\phi_{1,0,0}(\mathbf{x})$ or $\phi_{2,1,1}(\mathbf{x})$ with probabilities 1/4 and 3/4 respectively (we use the notation $\phi_{n\ell m}$). These probabilities are not correlated with the total momentum.

(a) Write an expression for $\psi(\mathbf{X}, \mathbf{x})$ ignoring the overall phase but including arbitrary constant phase factors where possible.

(b) What is the expectation value for the total energy?

6. **Virial Theorem and applications** [15 points]

   (a) Consider any time independent operator Ω and the time derivative of its expectation value, which is given by

   $$i\hbar \frac{d}{dt}\langle\Omega\rangle \;=\; \big\langle [\,\Omega\,, H\,]\big\rangle\,,$$

   where H is the Hamiltonian. Explain carefully why the right-hand side vanishes if the system is in a stationary state.

   (b) Now take $\Omega = \mathbf{r}\cdot\mathbf{p}$ and show that for any stationary state of the hydrogen atom Hamiltonian the following relation holds

   $$\langle T\rangle \;=\; -\tfrac{1}{2}\langle V\rangle\,.$$

   Here T is the kinetic energy operator and V is the potential energy operator.

   (c) For any hydrogen atom eigenstate write $\langle T\rangle = \frac{1}{2}m\langle v^2\rangle$, where m is quite accurately the mass of the electron. Express the ratio

   $$\frac{\sqrt{\langle v^2\rangle}}{c}$$

   in terms of the fine structure constant $\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}$ and the principal quantum number n. Is the electron relativistic? Give the corresponding results for the ground state when the nucleus has Z protons.

   (d) What is $\left\langle\frac{1}{r}\right\rangle$ in a general energy eigenstate of the hydrogen atom?

7. **Exercises on hydrogen atom and some generalizations** [10 points]

   (a) Find $\langle r\rangle$ and $\langle r^2\rangle$ in the ground state of hydrogen. What is the most probable value of r in the ground state?

   (b) Assume that the nucleus of hydrogen has radius one femtometer. Calculate the probability that the ground state electron is found inside the nucleus. Make approximations to simplify your work and still get a very accurate answer!

   (c) Positronium is a bound state of an electron and a positron (equal mass particles!). What are the energy levels? How does the size of positronium compare with the size of a hydrogen atom?