8.04: Quantum Mechanics — Professor Allan Adams
Massachusetts Institute of Technology — Tuesday March 19

Optional Problems on the Harmonic Oscillator

1. **Coherent States**

Consider a state φ_α which is an eigenstate of the annihilation operator

$$\hat{a}\,\varphi_\alpha = \alpha\,\varphi_\alpha \ ,$$

with α a complex number (see next page for a discussion of $\hat{a}$ and $\hat{a}^\dagger$). Such a state is called a "coherent state". Coherent states turn out to be the closest quantum analog of classical states with well-defined amplitudes and phases, and are extremely important in *e.g.* the quantum mechanical description of laser light, radio cavities, Bose-Einstein condensates, and just about everything else that's both macroscopic and quantum.

In this problem we'll explore some of the most basic features of coherent states.

(a) Show that

$$\hat{a}\left(\hat{a}^\dagger\right)^n \phi_0 = n\left(\hat{a}^\dagger\right)^{n-1}\phi_0 \ .$$

(b) Show that a coherent state φ_α can be written in the form,

$$\varphi_\alpha = C e^{\alpha\,\hat{a}^\dagger}\phi_0 \, ,$$

where C is a normalization constant and $e^{\alpha\,\hat{a}^\dagger}$ is defined via Taylor series.

(c) Calculate[1] C.

(d) Using your above results, express the coherent state φ_α as a superposition of the normalized energy eigenstates ϕ_n and calculate the probability of finding the coherent state in the n^{th} eigenstate.

(e) Calculate the average excitation level (ie $\langle \hat{N} \rangle$) in the coherent state φ_α. How does the energy expectation value depend on α?

(f) Which, if any, of the energy eigenstates of the HO are coherent states? To what value of α do they correspond?

(g) Compute Δx and Δp in the coherent state φ_α and verify that a coherent state is a minimal uncertainty wavepacket. *Aside: This is an important sense in which coherent states are as close to classical as it is possible to get. The next (bonus) problem will explore another way that coherent states behave classically.*

[1]Hint: Use the Baker-Campbell-Hausdorff formula which says that, for any two operators $\hat{X}$ and $\hat{Y}$, $e^{\hat{X}}e^{\hat{Y}} = e^{\hat{Y}}e^{\hat{X}}e^{[\hat{X},\hat{Y}]}$ so long as the commutator $[\hat{X},\hat{Y}]$ itself commutes with $\hat{X}$ and $\hat{Y}$, eg $[\hat{X},[\hat{X},\hat{Y}]] = 0$.

2. **Harmonic Oscillators Oscillate Harmonically II**

Consider a particle of mass m in a harmonic oscillator potential with wave function

$$\psi(x,0) = \left(\frac{1}{\pi\beta^2}\right)^{1/4} e^{-\frac{1}{2\beta^2}(x-x_o)^2}$$

where $\beta^2 = \frac{\hbar}{m\omega_o}$. This is nothing but the ground state wavefunction displaced from its equilibrium position, x=0, to x=x_o (think of a stretched spring). When $x_o \neq 0$, this state is *not* an eigenstate of the harmonic oscillator (think symmetry) and will thus evolve non-trivially in time. In this problem we will study its evolution.

(a) Act on the initial state with the annihilation operator (re-expressed in terms of $\hat{x}$ and $\hat{p}$). What kind of a state is this? *Hint: look back at the last problem...*

(b) Write down a formal expression for $\psi(x,t)$ at a subsequent time $t > 0$ as a superposition of energy eigenstates.

(c) Using the form of the energy eigenstates, $\phi_n(x) = \mathcal{N}_n H_n(x/\beta) e^{-\frac{1}{2\beta^2}x^2}$, show [2] that the coefficients of this expansion $c_n \equiv (\phi_n|\psi)$ take the form,

$$c_n = \frac{1}{\sqrt{n!}}\left(\frac{x_o}{\sqrt{2}\beta}\right)^n e^{-\frac{1}{4\beta^2}x_o^2}$$

[2]Hint: You can do this in many ways, including brute force, but a particularly efficient and illuminating way involves the idea of a *generating function*, as follows. Consider the function

$$Z(u,s) = e^{-s^2+2su} \; .$$

Performing a taylor expansion in s gives,

$$Z(u,s) = \sum_{n=0}^{\infty} \frac{H_n(u)}{n!} s^n \; .$$

where the $H_n(u)$ are the Hermite polynomials (to check, compare the first few terms in the taylor expansion to the table from lecture). Taking k derivatives of $Z(u,s)$ with respect to s then setting s to 0 thus gives,

$$H_k(u) = \left(\frac{\partial^k}{\partial s^k} Z(u,s)\right)\bigg|_{s=0} \; .$$

$Z(u,s)$ is called a *generating function* for the $H_n(u)$. The beauty of the generating function is that many things you'd have to do one at a time with the H_n (say, integrate them against a gaussian...) can be done all at once by using Z to get a generating function for your quantities of interest. If you play around a little bit, you should be able to use $Z(u,s)$ to build a generating function $C(u,s)$ for c_n and derive the stated result.

(d) Substitute this c_n into your formal expression for $\psi(x,t)$ and square to show that

$$\mathbb{P}(x,t) = \frac{1}{\sqrt{\pi}\beta}\, e^{-\frac{1}{\beta^2}(x - x_o \cos(\omega_o t))^2}$$

Our initially displaced wavepacket evolves in time with fixed *shape* but with its *center* oscillating around the minimum with an amplitude x_o and frequency ω_o!

Hint: Take advantage of the generating function $Z(x,s)$!

3. **More on Coherent States, and a little Squeezing**

The "coherent states" φ_α of a harmonic oscillator of mass m and frequency ω_o, which we studied above, are the eigenfunctions of the lowering operator $\hat{a}$,

$$\hat{a}\,\varphi_\alpha = \alpha\,\varphi_\alpha\,.$$

In this problem we'll illuminate the physics of these states by bringing to bear all of the operator tools we've built so far: the non-Hermitian raising and lowering operators $\hat{a}^\dagger$ and $\hat{a}$, the Hermitian Number and Energy operators $\hat{N}{=}\hat{a}^\dagger\hat{a}$ and $\hat{E}{=}\hbar\omega_o(\hat{N}+\frac{1}{2})$, the Unitary symmetry operators $\hat{T}_L$ and $\hat{B}_q$, and the Unitary time-propagator $\hat{U}_t$. *Note: At no point in this problem will you need the functional form of any wavefunctions. Everything will follow from the various operators and their commutation relations.*

(a) To begin, verify the following commutation relations (with γ a complex number):

$$\left[a,\hat{T}_L\right] = \frac{L}{\sqrt{2}\beta}\hat{T}_L\,, \qquad \left[a,\hat{B}_q\right] = \frac{iq\beta}{\sqrt{2}\hbar}\hat{B}_q\,, \quad \hat{a}\,e^{\gamma\hat{N}} = e^{\gamma}e^{\gamma\hat{N}}\hat{a}\,, \qquad \hat{a}^\dagger\,e^{\gamma\hat{N}} = e^{-\gamma}e^{\gamma\hat{N}}\hat{a}^\dagger$$

Physically, what do the first two commutators say about the set of coherent states?

(b) Consider the state

$$\phi_{x_op_o} = \hat{B}_{p_o}\hat{T}_{x_o}\varphi_0$$

where φ_0 is the ground state, $\hat{a}\,\varphi_0 = 0$. Deduce $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ in this state. How does the probability distribution differ from that of the ground state? Experimentally, if you are handed a harmonic oscillator in its ground state, how might you force the system into the state $\phi_{x_op_o}$?

(c) Using the commutators above, verify that $\phi_{x_op_o}$ is a coherent state, $\phi_{x_op_o} \equiv \varphi_{\alpha_o}$, and determine the corresponding eigenvalue α_o. Use your results to give a physical interpretation for a generic coherent state φ_α with $\hat{a}$-eigenvalue α.[3]

(d) Using the propagator $\hat{U}_t$ for the harmonic oscillator (cf problem 2e), and given the initial condition $\psi_{x_op_o}(x,0){=}\phi_{x_op_o}(x)$, we can formally write $\psi_{x_op_o}(x,t)$ as,

$$\psi_{x_op_o}(x,t) \;=\; \hat{U}_t\,\phi_{x_op_o}\,.$$

Show that $\psi_{x_op_o}(x,t)$ is again a coherent state with eigenvalue $\alpha_t = \alpha_o\,e^{-i\omega t}$.

(e) What are the expectation values $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ at time t? How do the uncertainties Δx and Δp vary with time t? How does the probability distribution $\mathbb{P}(x,t)$ vary with time? *Hint: No computations should be required!*

[3] Alternatively, use the Baker-Campbell-Hausdorff formula to directly show that $\phi_{x_op_o}{=}Ce^{\alpha_o\hat{a}^\dagger}\varphi_0$, where α_o is the same value you computed in part (c). Is $\phi_{x_op_o}$ is properly normalized?

(f) Now suppose we put our system in the state φ_{α_o} and then immediately *squeeze* the potential so that the frequency of the oscillator increases from ω_o to $\omega = s\,\omega_o$. Since the frequency appears in the definitions of $\hat{a}$ and $\hat{a}^\dagger$, the state φ_{α_o} will not be a coherent state of the new potential. However, it *is* an eigenstate of a linear combination of the raising and lowering operators of the new potential, $\hat{a}_s$ and $\hat{a}_s^\dagger$. Find the linear combination $\hat{b}_o = \mu_o \hat{a}_s + \nu_o \hat{a}_s^\dagger$ of whom the "squeezed state" φ_{α_o} is an eigenfunction with eigenvalue α_o. Observe that $|\mu_o|^2 - |\nu_o|^2 = 1$.

Aside: In general, "squeezed states" Φ_β *are eigenstates of linear combinations of* $\hat{a}$ *and* $\hat{a}^\dagger$*,*

$$\hat{b}\,\Phi_\beta = \beta\,\Phi_\beta\,, \qquad \hat{b} = \mu\,\hat{a} + \nu\,\hat{a}^\dagger\,,$$

where μ *and* ν *are complex numbers satisfying* $|\mu|^2 - |\nu|^2 = 1$ *so that* $\left[\hat{b}, \hat{b}^\dagger\right] = 1$.

(g) Using the propagator $\hat{U}_t^s$ for the squeezed potential, show the time-evolved state $\hat{U}_t^s\,\varphi_{\alpha_o}$ is a squeezed state with the same eigenvalue under a different operator,

$$\hat{b}_t\,\hat{U}_t^s \varphi_{\alpha_o} = \alpha_o\,\hat{U}_t^s \varphi_{\alpha_o}\,, \qquad \hat{b}_t = \mu_t \hat{a} + \nu_t \hat{a}^\dagger\,.$$

How do μ_t and ν_t depend on time?

(h) Determine $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$, as well as the widths Δx and Δp, in the state $\hat{U}_t^s \varphi_\alpha$. What has squeezing done to Δx and Δp? How do Δx and Δp evolve in time? Sketch the trajectory of $\langle x \rangle$ and $\langle p \rangle$ as a function of time, then indicate the widths Δx and Δp by using them to define an ellipse centered around the expectation values at each moment in time. Indicate the effect of squeezing in your sketch.

Aside: Lest this seem like so much formal manipulation, you should know that squeezed states of the quantum harmonic oscillator turn out to be exceedingly important experimental tools, used in everything from atomic physics to the detection of gravitational waves. Seriously! If you're curious about this, the world's expert is MIT's own Prof. Nergis Mavalvala, an excellent physicist and a powerful quantum mechanic.