8.04: Quantum Mechanics
Massachusetts Institute of Technology

Professor Allan Adams
Friday May 24, 2012

Final Exam

Last Name: ______________________________

First Name: ______________________________

Check	Recitation	Instructor	Time
	R01	Barton Zwiebach	10:00
	R02	Barton Zwiebach	11:00
	R03	Matt Evans	3:00
	R04	Matt Evans	4:00

Instructions:
Show all work. All work must be done in this exam packet. Do not remove any pages.
This is a closed book exam – books, notes, phones, calculators etc are **not allowed**.
You have 3 hours to solve the problems. Exams will be collected at 12:00pm.

Problem	Max Points	Score	Grader
1	60		
2	15		
3	20		
4	30		
5	35		
6	40		
Total	200		

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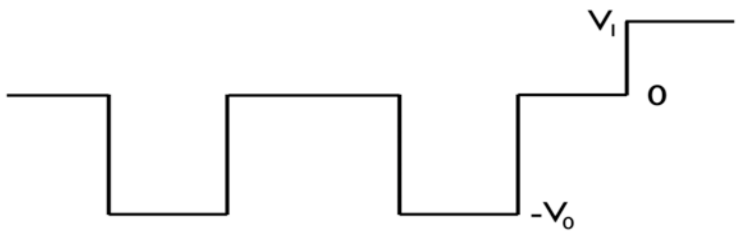
1. **(60 points) Short Answer**

Consider the potential below. Assume that the potential vanishes to the left of the region shown, and remains equal to V_1 to the right. Let E_n and ϕ_n denote the energy eigenvalues and eigenfunctions, respectively.



(a) (5 Points) Circle each correct statement valid for all eigenfunctions:

ϕ_n may be chosen to be real

ϕ_n may be chosen to be even or odd

(b) (5 Points) For $-V_0 < E < 0$, circle each correct statement:

E_n discrete

E_n degenerate

ϕ_n normalizable

(c) (5 Points) For $0 < E < V_1$, circle each correct statement:

E_n discrete

E_n degenerate

ϕ_n normalizable

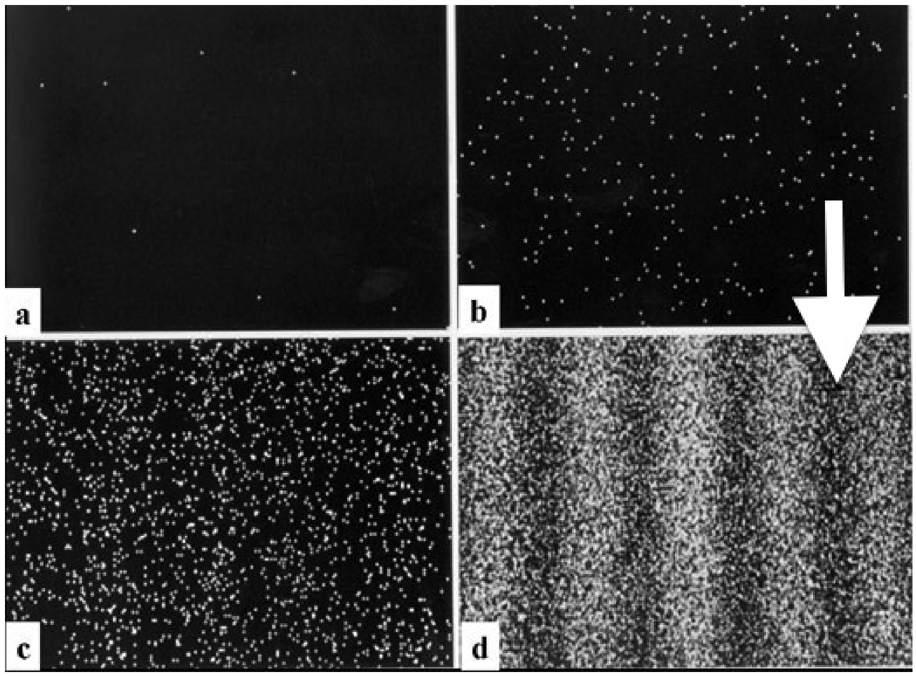
(d) (5 Points) For $V_1 < E$, circle each correct statement:

E_n discrete

E_n degenerate

ϕ_n normalizable

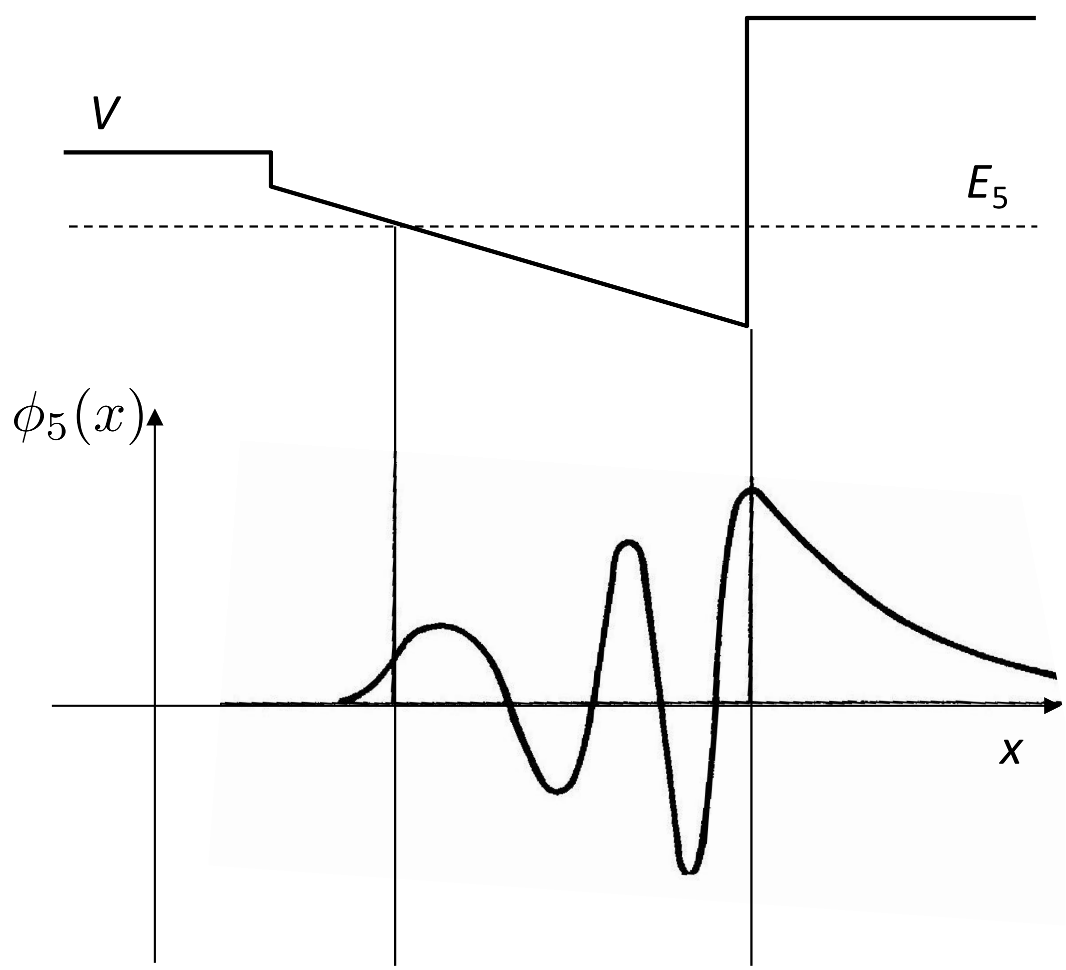
(e) (5 Points) Below are two-slit interference patterns generated by sending 8, 270, 2,000 and 160,000 electrons with de Broglie wavelength λ through two slits over the course of 20 minutes (stills from the Hitachi video shown in class).



Consider image (d). If the maximum on the left is equidistant from both slits, how much further from the first slit than from the second slit is the minimum at the right of the image (indicated by the white arrow)? Circle one:

0.5λ $\quad$ 1.5λ $\quad$ 2λ $\quad$ 3.5λ $\quad$ 4λ

(f) (5 Points) The curve in the figure below is alleged to be the plot of a computer-calculated wave function for the 5th excited state of a particle in the diagrammed one-dimensional potential well. By means of arrows and labels, indicate the way or ways in which the plot fails to be even qualitatively correct.

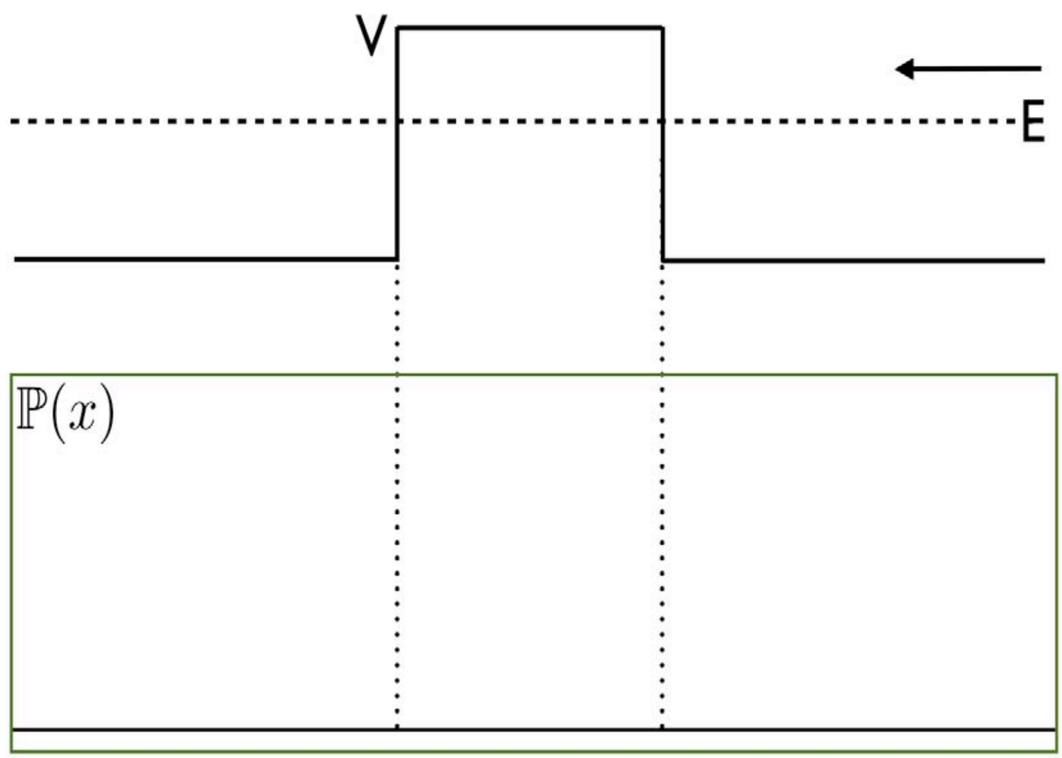


A particle of mass m moves freely along a circle of length L. Let x denote the position along the circle, with $0 \leq x < L$.

(g) (5 Points) A textbook on my desk claims that the wavefunction may satisfy the boundary condition $\psi(L) = -\psi(0)$. Is this possible? Justify your answer.

(h) (5 Points) Use the physical boundary condition to determine the energy eigenstates and eigenvalues for this system. What is the degeneracy of each energy eigenvalue?

(i) (5 Points) Sketch the probability density for the energy eigenfunction corresponding to a particle incident *from the right* on the potential below, with energy $0 < E < V$ as indicated. (You may find it useful to write out the general form of the wavefunction in each region)



Consider two non-interacting particles which are placed in the same potential, $V(x)$, whose single-particle energy eigenfunctions are $\phi_n(x)$ with non-degenerate energies E_n. Let $\Psi(x_1, x_2)$ denote the wavefunction for the two-particle system.

(j) (5 points) Identify a Ψ which is possible if the particles are identical **bosons**.

(k) (5 points) Identify a Ψ which is possible if the particles are identical **fermions**.

(l) (5 points) Identify a Ψ which is only possible if the particles are **distinguishable**.

2. **(15 Points) Diamonds are a Condensed Matter Physicist's Best Friend**

   (a) (10 Points) Diamond has a band gap E_g equal to $5.5eV$ (at standard temperature and pressure). Use this number to derive a rough estimate of the lattice spacing, a, of the diamond lattice. Do you expect the true lattice spacing to be larger or smaller than your estimate?

(b) (5 Points) What is the minimum wavelength at which a diamond in a jewelry store is opaque? How does this wavelength depend on the size of the diamond?

3. **(20 Points) Degeneracies in a 2d Box**

   Consider a particle of mass m in $2d$ confined to a square box of side lengths $L_x = L_y = L$ by an infinite potential. Let the potential inside the box vanish.

   (a) (5 Points) What are the energy eigenvalues in this system? State the degeneracies of the four lowest energy eigenvalues.

   (b) (5 Points) What is the origin of the degeneracies you found in part (a)?

(c) (5 Points) Suggest a modification of the side lengths which will lift all degeneracy.

(d) (5 Points) Suggest a modification of the side lengths which will reduce, but not eliminate, the number of degenerate energies.

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4. **(30 Points) Fun with Hydrogen**

At time $t = 0$, an electron in a hydrogen atom is in the state

$$\psi(\vec{r}, 0) = A\left[3i\phi_{100}(\vec{r}) - 4\phi_{211}(\vec{r}) - i\phi_{210}(\vec{r}) + \sqrt{10}\phi_{21-1}(\vec{r})\right]$$

where $\phi_{n\ell m}$ are the properly-normalized energy eigenstates, with $n = 1$ denoting the ground state.

(a) (2 Points) Determine A.

(b) (4 Points) What is the wavefunction at time t, ie $\psi(\vec{r}, t)$?

(c) (4 Points) What is the expectation value $\langle E \rangle$ at $t = 0$? (In terms of E_1)

(d) (4 Points) What is the expectation value $\langle L^2 \rangle$ at $t = 0$?

(e) (4 Points) What is the expectation value $\langle L_z \rangle$ at $t = 0$?

(f) (4 Points) Which of $\langle E \rangle$, $\langle L_z \rangle$, $\langle L^2 \rangle$ and $\langle \vec{r} \rangle$ vary with time in this state?

(g) (4 Points) What is the shortest time $t_o > 0$ at which the spatial dependence of the *probability density* $\mathbb{P}(\vec{x}, t)$ returns to what it was at $t = 0$?

(h) (4 Points) Suppose that a measurement of L_z at $t = 0$ yields $\hbar$. After this measurement, what is the properly normalized wavefunction, $\psi(\vec{r}, t)$?

5. **(35 Points) An Imperfect Mirror**

Consider the following attractive potential on the half-line $x > 0$,

$$V(x) = \left\{ \begin{array}{cc} \infty & x < 0 \\ -V_o & 0 < x < L \\ 0 & L < x \end{array} \right\}$$

with V_0 a positive constant with units of energy. Define the dimensionless constant z_0 that describes the strength of the potential:

$$z_0 \equiv \frac{2mL^2V_0}{\hbar^2}\,.$$

(a) (10 Points) Suppose the potential has precisely two bound states. Sketch the corresponding wavefunctions. How does the higher energy bound state look when its energy is close to zero? Do another sketch for that!

(b) (5 Points) Assume the second bound state is at threshold (it has energy just infinitesimally below zero). What is the value of z_0 for this to occur?

(c) (5 Points) What range of values of z_0 is consistent with precisely two bound states?

Consider now an energy eigenstate of positive energy $E > 0$ represented by a wave incoming from the right and a reflected wave so that, for $x > L$, we have:

$$\psi(x) = Ce^{ikx} + De^{-ikx}\,, \quad x > L\,, \qquad k^2 = \frac{2mE}{\hbar^2}\,.$$

The purpose of the following steps is to calculate the reflection coefficient $r \equiv \frac{C}{D}$.

(d) (10 Points) Choose an ansatz for $\psi(x)$ in the region $0<x<L$ and use the relevant boundary conditions to calculate r in terms of k, L and the constant k' defined as

$$k'^2 = \frac{2m(E+V_0)}{\hbar^2}\,.$$

Your answer should take the form

$$r = -e^{(\ldots)}\Big(\frac{1+\ldots}{1-\ldots}\Big)$$

where the dots represent quantities to be determined (not all equal!).

(e) (5 Points) What *should* your answer for r reduce to when $V_0 = 0$? Does your answer above do that? Show it.

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6. **(40 Points) Three Passes at the 1d Harmonic Oscillator**

   Consider a particle of mass m in a 1d harmonic oscilator potential with frequency ω,

$$\hat{E} = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{x}^2 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

   (a) (15 Points) At time $t = 0$, the particle has equal probability of being measured to have energy E_n and E_m, with $n > m$. Specify any state, $\psi(t)$, which has this property and compute the velocity of the particle as a function of time, where the velocity is defined by,

$$v(t) = \frac{d}{dt}\langle x\rangle$$

   How does your result depend on the values of m and n?

   Note: You do not need the functional forms of the $\phi_n(x)$ to answer this question!

(b) (10 Points) Use what you know about the quantum harmonic oscillator to deduce the allowed energies for a particle in the *half*–harmonic oscillator potential,

$$V(x) = \left\{ \begin{array}{cc} \frac{1}{2}\mu\omega^2 x^2 & x > 0 \\ \infty & x < 0 \end{array} \right\}$$

Be sure to explain your answer.

Note: This problem requires careful thought but very little computation.

(c) (15 Points) Consider a novel harmonic oscillator in which the energy operator takes the form,

$$\hat{E}_F = \hbar\omega\,\hat{\alpha}^\dagger\hat{\alpha}\,,$$

where $\hat{\alpha}$ and $\hat{\alpha}^\dagger$ satisfy the *anti-commutation* relations

$$\hat{\alpha}\hat{\alpha}^\dagger + \hat{\alpha}^\dagger\hat{\alpha} = 1\,, \qquad \hat{\alpha}^\dagger\hat{\alpha}^\dagger = 0\,, \qquad \hat{\alpha}\hat{\alpha} = 0\,.$$

Use these anti-commutation relations to show the following:

i. Show that $(\hat{E}_F)^2 = \hbar\omega\hat{E}_F$. What are the possible eigenvalues of $\hat{E}_F$?

ii. Show that there must be a state annihilated by $\hat{\alpha}$. Call this state ϕ_0.

iii. Show that ϕ_0 cannot be annihilated by $\hat{\alpha}^\dagger$, i.e. that $\hat{\alpha}^\dagger \phi_0 \neq 0$. Let $\phi_1 \equiv \hat{\alpha}^\dagger \phi_0$.

iv. Show that $\hat{\alpha}\, \phi_1 = \phi_0$ and that $\hat{\alpha}^\dagger \phi_1 = 0$.

v. Verify that ϕ_0 and ϕ_1 are eigenfunctions of $\hat{E}_F$, and find their eigenvalues.

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