Problem Set 5	**Solutions**
8.04 Spring 2013	**March 12, 2013**

Problem 1. (10 points) The Probability Current

We wish to prove that

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t). \tag{1}$$

Since $P_{ab}(t)$ is the probability of finding the particle in the range $a < x < b$ at time t it is mathematically equal to

$$P_{ab}(t) = \int_a^b |\psi(x,t)|^2 dx = \int_a^b \psi^*(x,t)\psi(x,t)dx. \tag{2}$$

Its time derivative is therefore given by

$$\frac{dP_{ab}}{dt} = \frac{d}{dt}\int_a^b \psi^*(x,t)\psi(x,t)dx = \int_a^b \frac{\partial}{\partial t}\left[\psi^*(x,t)\psi(x,t)\right]dx = \int_a^b \left[\psi^*\frac{\partial\psi}{\partial t} + \frac{\partial\psi^*}{\partial t}\psi\right]dx, \tag{3}$$

where we were allowed to take the time derivative inside the integral because the integral is in *time* whereas the integral is over *space*. Note, though, that the total derivative became a partial derivative when we took it inside the integral, because whereas $\int_a^b \psi^*(x,t)\psi(x,t)dx$ is a function of t only (since x has already been integrated over), ψ is in general a function of both x and t, and we only want to take its time derivative.

We now make use of the fact that regardless of what the wavefunction $\psi(x,t)$ happens to be, it must obey the Schrödinger equation:

$$i\hbar\frac{\partial\psi(x,t)}{dt} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) \tag{4a}$$

$$-i\hbar\frac{\partial\psi^*(x,t)}{dt} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi^*(x,t)}{\partial x^2} + V(x)\psi^*(x,t) \tag{4b}$$

We can thus eliminate the partial derivatives in Equation 3:

$$\frac{dP_{ab}}{dt} = \frac{1}{i\hbar}\int_a^b \left[\psi^*\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi\right) - \left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} + V\psi^*\right)\psi\right]dx \tag{5a}$$

$$= \frac{1}{i\hbar}\int_a^b \left[-\frac{\hbar^2}{2m}\left(\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right) + \left(\psi^*V\psi - V\psi^*\psi\right)\right]dx \tag{5b}$$

$$= \frac{i\hbar}{2m}\int_a^b \left(\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right)dx, \tag{5c}$$

where in the last equality we used the fact that $\psi^*V\psi = V\psi^*\psi$ because in position space the potential energy operator commutes with both ψ and ψ^*. To proceed, we integrate the two remaining terms by parts. Recall that

$$\int_a^b f\frac{\partial g}{\partial x}dx = fg\Big|_a^b - \int_a^b \frac{\partial f}{\partial x}g\,dx. \tag{6}$$

This means

$$\frac{dP_{ab}}{dt} = \frac{i}{2m}\int_a^b \left(\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right)dx = \frac{i}{2m}\int_a^b \left[\psi^*\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right) - \psi\frac{\partial}{\partial x}\left(\frac{\partial\psi^*}{\partial x}\right)\right]dx \quad (7a)$$

$$= \frac{i}{2m}\left[\left(\psi^*\frac{\partial\psi}{\partial x}\right)\Big|_a^b - \left(\psi\frac{\partial\psi^*}{\partial x}\right)\Big|_a^b - \int_a^b \frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}dx + \int_a^b \frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x}dx\right] \quad (7b)$$

$$= \frac{i}{2m}\left[\left(\psi^*\frac{\partial\psi}{\partial x}\right)\Big|_a^b - \left(\psi\frac{\partial\psi^*}{\partial x}\right)\Big|_a^b\right] \quad (7c)$$

$$= \frac{i}{2m}\left[\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)\Big|_{x=a} - \left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)\Big|_{x=b}\right], \quad (7d)$$

so if we let

$$J(x,t) \equiv \frac{i}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right), \quad (8)$$

then

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t), \quad (9)$$

which is what we were trying to prove. From this equation, we can see that the units of $J(x,t)$ must be $(\text{time})^{-1}$, because P_{ab} is a probability and is therefore a pure number, and t has units of time.

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Problem 2. (20 points) Visual Observation of a Quantum Harmonic Oscillator

(a) **(5 points)** The energy of a classical harmonic oscillator is given by

$$E = \frac{1}{2} m\omega_0^2 A^2, \tag{10}$$

where ω_0 is the angular frequency, m is the mass, and A is the amplitude of the oscillation. The quantum harmonic oscillator[1], on the other hand, has energy

$$E_n = \hbar\omega_0 \left(n + \frac{1}{2} \right). \tag{11}$$

Equating these expressions and rearranging gives

$$n \approx \frac{\frac{1}{2} m\omega_0^2 A^2}{\ \omega_0} = \frac{1}{2} \frac{m\omega_0}{\ } A^2 = \frac{\pi m \nu_0}{\ } A^2, \tag{12}$$

where we have converted to using the frequency $\nu_0 = \omega_0/2\pi$ instead of the angular frequency ω_0, and have assumed that n is so large that $n + \frac{1}{2} \approx n$ (we can check our result against this assumption later). Plugging in $m = 10^{-12}\,\mathrm{g}$, $\nu_0 = 10^3\,\mathrm{Hz}$, and $A = 10^{-3}\,\mathrm{cm}$, we get

$$n \approx 3 \times 10^{12}. \tag{13}$$

From this result we see that we were justified in neglecting the 1/2 term in $n + \frac{1}{2}$.

(b) **(5 points)** If this oscillator were in its ground state, its energy would be

$$E_0 = \frac{1}{2} \hbar\omega_0 = \pi \ \ \nu_0 = 2.1 \times 10^{-12}\,\mathrm{eV}, \tag{14}$$

where once again $\nu_0 = 10^3\,\mathrm{Hz}$. This is smaller than the average thermal energy of air molecules (25 meV) by a factor of

$$\frac{E_{air}}{E_0} = \frac{25 \times 10^{-3}\mathrm{eV}}{2.1 \times 10^{-12}\,\mathrm{eV}} \approx 10^{10}. \tag{15}$$

(c) **(5 points)** We can perform the same manipulations as we did in part (a), except this time we *cannot* neglect the 1/2 term in the oscillator's energy, since we are dealing with the ground state:

$$E_0 = \frac{1}{2} \ \ \omega_0 = \frac{1}{2} m\omega_0^2 A^2 \quad \Rightarrow \quad A = \sqrt{\frac{\hbar}{m\omega_0}} = \sqrt{\frac{\hbar}{2\pi m \nu_0}}. \tag{16}$$

[1] There is really no difference between a classical harmonic oscillator and a quantum one — the Universe is governed by quantum mechanics, so in principle all oscillators are quantum mechanical. The only difference is whether or not the quantum effects are obvious.

Note that this is width of the Gaussian ground state wavefunction. Numerically, this comes out to be $A = 0.00401\,\mathrm{nm}$, which is much smaller than the wavelength of visible light:

$$\frac{\lambda}{A} = \frac{400\,\mathrm{nm}}{0.00401\,\mathrm{nm}} \approx 10^5. \tag{17}$$

(d) **(5 points)** I would *not* recommend award of a grant to carry out this research. From part (a), we can see that the experimenter's proposed system corresponds to a high quantum number state, which means the experiment is unlikely to really probe quantum mechanical effects. At best, it would be able to investigate the approach of the quantum system to the classical limit, but even this is unlikely, given the results of (b) and (c). In part (b), we saw that air molecules possess energies that are much larger than the spacings between energy levels of the quantum harmonic oscillator. Thus, unless the experimenter can perform his/her experiment in a perfect vacuum (something that *nobody* can achieve), the air molecules will likely interact with the system and cause the oscillator to make transitions to higher energy levels. Finally, the calculation in part (c) shows that using visible light to probe the system is not practical. Even if one neglects the collapse of the wavefunction caused by measuring the system with light, the fact that the relevant spatial scales ($\sim 0.004\,\mathrm{nm}$) are so much smaller the wavelength of the light means that the light will essentially ignore features on those scales. This is why, for instance, mirrors for telescopes that operate in the visible wavelength band have to be grounded to such high precision, whereas those for microwave telescopes are about as smooth as a nice piece of cardboard.

Problem 3. (30 points) Harmonic Oscillators Oscillate Harmonically

(a) **(4 points)** Our wavefunction is initially

$$\psi(x,0) = \frac{1}{\sqrt{2}}[\phi_0(x) + i\phi_1(x)], \tag{18}$$

where ϕ_0 and ϕ_1 are the normalized eigenstates for the ground and first excited states of the harmonic oscillator respectively. The eigenstates evolve in time in the usual fashion (phase factor with angular frequency equal to $-E_n/\hbar$), so the principle of superposition tells us that

$$\psi(x,t) = \frac{1}{\sqrt{2}}\left[e^{-iE_0t/}\ \phi_0(x) + ie^{-iE_1t/}\ \phi_1(x)\right] = \frac{1}{\sqrt{2}}\left[e^{-i\omega_0t/2}\phi_0(x) + ie^{-i3\omega_0t/2}\phi_1(x)\right], \tag{19}$$

where in the last step we used the fact that the nth excited state of a harmonic oscillator has energy $E_n = \hbar\omega_0(n+\frac{1}{2})$. The probability distribution $|\psi(x,t)|^2$ is given by

$$\begin{aligned}
|\psi(x,t)|^2 \equiv \psi^*\psi &= \frac{1}{2}\left[e^{-i\omega_0t/2}\phi_0(x) + ie^{-i3\omega_0t/2}\phi_1(x)\right]^*\left[e^{-i\omega_0t/2}\phi_0(x) + ie^{-i3\omega_0t/2}\phi_1(x)\right] && \text{(20a)}\\
&= \frac{1}{2}\left[e^{i\omega_0t/2}\phi_0^*(x) - ie^{i3\omega_0t/2}\phi_1^*(x)\right]\left[e^{-i\omega_0t/2}\phi_0(x) + ie^{-i3\omega_0t/2}\phi_1(x)\right] && \text{(20b)}\\
&= \frac{1}{2}\left[|\phi_0(x)|^2 + |\phi_1(x)|^2 - ie^{i\omega_0t}\phi_1^*\phi_0 + ie^{-i\omega_0t}\phi_0^*\phi_1\right] && \text{(20c)}\\
&= \frac{1}{2}\left[|\phi_0(x)|^2 + |\phi_1(x)|^2 + i\phi_0\phi_1\left(e^{-i\omega_0t} - e^{i\omega_0t}\right)\right] && \text{(20d)}\\
&= \frac{1}{2}\left[|\phi_0(x)|^2 + |\phi_1(x)|^2 + 2\phi_0\phi_1\sin\omega_0t\right], && \text{(20e)}
\end{aligned}$$

where in the second last step we took advantage of the fact that energy eigenstates can be taken to be real (see Problem Set 3), so $\phi_0^* = \phi_0$ and $\phi_1^* = \phi_1$.

(b) **(10 points)** The expectation value $\langle\hat{x}\rangle$ is defined in the usual fashion:

$$\langle\hat{x}\rangle = \int_{-\infty}^{\infty}\psi^*\hat{x}\psi dx = \int_{-\infty}^{\infty}x|\psi(x,t)|^2dx = \frac{1}{2}\left[\int_{-\infty}^{\infty}x|\phi_0(x)|^2dx + \int_{-\infty}^{\infty}x|\phi_1(x)|^2dx + 2\sin\omega_0t\int_{-\infty}^{\infty}x\phi_0\phi_1dx\right]. \tag{21}$$

To proceed, we need to know something about the form of ϕ_n, the nth excited energy eigenstate of the harmonic oscillator. As discussed in lecture, the general form of ϕ_n is given by

$$\phi_n(x) = \mathcal{N}_nH_n(x/a)e^{-\frac{x^2}{2a^2}}, \tag{22}$$

where H_n is the nth Hermite polynomial and $a \equiv \sqrt{\hbar/m\omega_0}$. Since the Hermite polynomials are either even or odd, the eigenstates are also either even and odd[2], which means that $|\phi_0(x)|^2$ and $|\phi_1(x)|^2$ are both even. In turn, $x|\phi_0(x)|^2$ and $x|\phi_1(x)|^2$ are odd, so

$$\int_{-\infty}^{\infty} x|\phi_0(x)|^2 dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} x|\phi_1(x)|^2 dx = 0, \tag{23}$$

since the integrals are over symmetric intervals. This leaves the last term, for which we will need the precise forms of the energy eigenstates:

$$\phi_0(x) = \left(\frac{m\omega_0}{\pi}\right)^{1/4} e^{-\frac{x^2}{2a^2}} \quad \text{and} \quad \phi_1(x) = \left(\frac{m\omega_0}{\pi}\right)^{1/4} \frac{x\sqrt{2}}{a} e^{-\frac{x^2}{2a^2}} \tag{24}$$

This gives

$$\langle \hat{x} \rangle = \sin\omega_0 t \int_{-\infty}^{\infty} x\phi_0\phi_1 dx = a\sqrt{\frac{2m\omega_0}{\pi}} \sin\omega_0 t \int_{-\infty}^{\infty} \left(\frac{x}{a}\right)^2 e^{-\left(\frac{x}{a}\right)^2} dx \tag{25a}$$

$$= a^2\sqrt{\frac{2m\omega_0}{\pi}} \sin\omega_0 t \int_{-\infty}^{\infty} u^2 e^{-u^2} du = a^2\sqrt{\frac{m\omega_0}{2}} \sin\omega_0 t = \sqrt{\frac{\hbar}{2m\omega_0}} \sin\omega_o t. \tag{25b}$$

The reader is encouraged to check that this expression has the right units. The amplitude of the oscillation is $2\sqrt{\frac{\hbar}{2m\omega_0}}$, and the angular frequency of the oscillation is ω_0.
It is instructive to see how to calculate $\langle \hat{x} \rangle$ also via the operator method. Taking the result from Problem 4 (e), we can express $\hat{x}$ in terms of $\hat{a}, \hat{a}^\dagger$:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}}(\hat{a} + \hat{a}^\dagger). \tag{26}$$

Thus

$$\langle \hat{x} \rangle = \sqrt{\frac{\hbar}{2m\omega_0}} \int dx \psi^*(\hat{a} + \hat{a}^\dagger)\psi =$$

$$= \sqrt{\frac{\hbar}{2m\omega_0}} \;\; \frac{1}{2}\int \phi_0(\hat{a} + \hat{a}^\dagger)\phi_0 + \frac{i}{2}e^{-i\omega_0 t}\int \phi_0(\hat{a} + \hat{a}^\dagger)\phi_1 +$$

$$-\frac{i}{2}e^{i\omega_0 t}\int \phi_1(\hat{a} + \hat{a}^\dagger)\phi_0 + \frac{1}{2}\int \phi_1(\hat{a} + \hat{a}^\dagger)\phi_1 \Bigg]. \tag{27}$$

Now, still from Problem 4 we know that, for any energy eigenstate ϕ_n,

$$\int \phi_n \hat{a} \phi_n = \int \phi_n \hat{a}^\dagger \phi_n = \int \phi_n \hat{a} \phi_{n-1} = \int \phi_{n-1} \hat{a}^\dagger \phi_n = 0, \tag{28}$$

and thus we have

$$\langle \hat{x} \rangle = \sqrt{\frac{\hbar}{2m\omega_0}} \left[\frac{i}{2}e^{-i\omega_0 t}\int \phi_0 \hat{a} \phi_1 - \frac{i}{2}e^{i\omega_0 t}\int \phi_1 \hat{a}^\dagger \phi_0\right] = \sqrt{\frac{\hbar}{2m\omega_0}} \sin\omega_0 t. \tag{29}$$

[2] Of course, we know from Problem Set 3 that we can *always* take the energy eigenstates to be even or odd, but it's nice to see it explicitly in Equation 22 as well.

(c) **(10 points)** For $\langle p \rangle$ we have

$$\langle \hat{p} \rangle = \int_{-\infty}^{\infty} \psi^* \hat{p} \psi dx = \int_{-\infty}^{\infty} \psi^* \frac{\hbar}{i} \frac{\partial}{\partial x} \psi dx \tag{30a}$$

$$= \frac{\hbar}{2i} \int_{-\infty}^{\infty} \left[e^{i\omega_0 t/2} \phi_0^*(x) - i e^{i3\omega_0 t/2} \phi_1^*(x) \right] \left[e^{-i\omega_0 t/2} \phi_0'(x) + i e^{-i3\omega_0 t/2} \phi_1'(x) \right] dx \tag{30b}$$

$$= \frac{\hbar}{2i} \int_{-\infty}^{\infty} \left[\phi_0 \phi_0' + \phi_1 \phi_1' - i e^{i\omega_0 t} \phi_1 \phi_0' + i e^{-i\omega_0 t} \phi_0 \phi_1' \right] dx, \tag{30c}$$

where we have once again used the fact that the energy eigenfunctions can be taken to be real. Let us examine this term-by-term. Since the derivative of an even function is odd (and vice versa[3]), terms like $\phi_n \phi_n'$ must be odd overall. We thus have

$$\int_{-\infty}^{\infty} \phi_0 \phi_0' dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \phi_1 \phi_1' dx = 0. \tag{31}$$

Plugging in the forms of the energy eigenfunctions gives

$$\langle \hat{p} \rangle = \frac{\hbar}{2i} \int_{-\infty}^{\infty} \left[-i e^{i\omega_0 t} \phi_1 \phi_0' + i e^{-i\omega_0 t} \phi_0 \phi_1' \right] dx \tag{32a}$$

$$= \frac{1}{2} \sqrt{\frac{m\hbar\omega_0}{\pi}} \int_{-\infty}^{\infty} \left[e^{i\omega_0 t} \frac{x^2 \sqrt{2}}{a^3} e^{-\frac{x^2}{a^2}} - e^{-i\omega_0 t} \frac{x^2 \sqrt{2}}{a^3} e^{-\frac{x^2}{a^2}} + e^{-i\omega_0 t} \sqrt{2} \frac{e^{-\frac{x^2}{a^2}}}{a} \right] dx \tag{32b}$$

$$= \frac{1}{2} \sqrt{\frac{m\hbar\omega_0}{\pi}} \int_{-\infty}^{\infty} \left[e^{i\omega_0 t} \sqrt{2} u^2 e^{-u^2} - e^{-i\omega_0 t} u^2 \sqrt{2} e^{-u^2} + e^{-i\omega_0 t} e^{-u^2} \sqrt{2} \right] du \tag{32c}$$

$$= \frac{1}{2} \sqrt{\frac{m\hbar\omega_0}{\pi}} \left[e^{i\omega_0 t} \frac{\sqrt{\pi}}{\sqrt{2}} - e^{-i\omega_0 t} \frac{\sqrt{\pi}}{\sqrt{2}} + e^{-i\omega_0 t} \sqrt{2\pi} \right] \tag{32d}$$

$$= \frac{1}{2} \sqrt{\frac{m\hbar\omega_0}{\pi}} \left[i \sin \omega_0 t \sqrt{2\pi} + (\cos \omega_0 t - i \sin \omega_0 t) 2\sqrt{\pi} \right] \tag{32e}$$

$$= \sqrt{\frac{m\hbar\omega_0}{2}} \cos \omega_0 t. \tag{32f}$$

Again, let's obtain the expression for $\langle \hat{p} \rangle$ also through the operator method. From Problem 4 (e), we know that

$$\hat{p} = -i \sqrt{\frac{m\hbar\omega_0}{2}} (\hat{a} - \hat{a}^\dagger), \tag{33}$$

and hence,

$$\langle \hat{p} \rangle = -i \sqrt{\frac{m \ \omega_0}{2}} \int dx \psi^* (\hat{a} - \hat{a}^\dagger) \psi. \tag{34}$$

[3]If you're not sure about this, try proving it from the definition of the derivative: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$. It may also help to compute the derivatives of the energy eigenstates of the harmonic oscillator explicitly. Finally, try drawing a sketch! A picture is worth 10^3 words, especially in physics.

Now, expanding like in Equation (27) and keeping into account Equation (28), we obtain

$$\langle \hat{p} \rangle = -i\sqrt{\frac{m\hbar\omega_0}{2}} \int \left[\frac{i}{2} e^{-i\omega_0 t} \int \phi_0 \hat{a} \phi_1 + \frac{i}{2} e^{i\omega_0 t} \int \phi_1 \hat{a}^\dagger \phi_0 \right] = \sqrt{\frac{m \ \omega_0}{2}} \cos \omega_0 t. \quad (35)$$

(d) **(6 points)** Any wavefunction can be expanded in terms of the energy eigenstates of the system:

$$\psi(x,t) = \sum_n c_n \phi_n(x) e^{-iE_n t/ \ } \ , \quad (36)$$

where E_n is the energy of the nth excited state. With the harmonic oscillator, we have

$$E_n = \ \omega_0 \left(n + \frac{1}{2} \right), \quad (37)$$

so the wavefunction takes the form

$$\psi(x,t) = e^{-i\omega_0 t/2} \sum_n c_n \phi_n(x) e^{-in\omega_0 t}. \quad (38)$$

Each term in the sum oscillates with period $T_n = \frac{2\pi}{n\omega_0}$, *i.e.* some integer fraction of the classical period $T = 2\pi/\omega_0$. For example, aside from the overall the $e^{-i\omega_0 t/2}$ in front, the first term is constant, the second term oscillates with period T, the third term oscillates with period $T/2$, and so on. Thus, after a time T all the phases will be aligned once again, except for phase in the prefactor, so that

$$\psi(x, t+T) = e^{-i\omega_0 T/2} \psi(x,t). \quad (39)$$

Taking the norm squared of both sides gives

$$|\psi(x,t+T)|^2 = |\psi(x,t)|^2, \quad (40)$$

which is a mathematical way of saying that the probability distribution returns to its original shape after period $T = 2\pi/\omega_0$.

The probability distribution is periodic because the difference between angular frequencies $(E_n - E_m)/\hbar$ are rational multiples of a common value. With the harmonic oscillator in particular, this arises because the energy levels are equally spaced apart (with $\Delta E = \ \omega_0$).
Consider instead a system whose first three energy levels are $E_0, \sqrt{2}E_0, 2E_0$, with associated eigenstates ϕ_0, ϕ_1, ϕ_2. Then the state

$$\psi(x,t) = \frac{1}{\sqrt{3}} \left(e^{-i\frac{E_0}{\hbar}t} \phi_0 + e^{-i\frac{\sqrt{2}E_0}{\hbar}t} \phi_0 + e^{-i\frac{2E_0}{\hbar}t} \phi_2 \right)$$

is not periodic in time, or more precisely there is no value of T for which

$$\phi(x,t) = e^{i\alpha} \psi(x, t+T). \quad (41)$$

To see this, let's rewrite ψ as

$$\psi(x,t) = \frac{e^{-i\frac{E_0}{\hbar}t}}{\sqrt{3}} \left(\phi_0 + e^{-i\frac{(\sqrt{2}-1)E_0}{\hbar}t}\phi_0 + e^{-i\frac{E_0}{\hbar}t}\phi_2 \right),$$

so that (41) is equivalent to

$$e^{-i(\sqrt{2}-1)\frac{E_0}{\hbar}T} = 1, \qquad \text{together with} \qquad e^{-i\frac{E_0}{\hbar}T} = 1,$$

which is equivalent to

$$(\sqrt{2}-1)T = 2\pi n\frac{\hbar}{E_0}, \qquad \text{together with} \qquad T = 2\pi m\frac{\hbar}{E_0}$$

for some integers n and m. Dividing one equation by the other we get

$$\sqrt{2}-1 = \frac{n}{m},$$

which has no solution.

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Problem 4. (40 points) Operators for the Harmonic Oscillator

(a) **(10 points)** First remember what it means when we take the Hermitian conjugate ("dagger") of an operator. The operator $\mathcal{O}^\dagger$ is defined by the following equation: where ψ_1 and ψ_2 are arbitrary wavefunctions that satisfy that usual requirements of continuity, normalizability, and so on. With this, the norm squared of our state is given by

$$(\tilde{\phi}_n, \tilde{\phi}_n) = \int_{-\infty}^{\infty} \tilde{\phi}_n^* \tilde{\phi}_n dx = \int_{-\infty}^{\infty} [(\hat{a}^\dagger)^n \phi_0]^* (\hat{a}^\dagger)^n \phi_0 dx = \int_{-\infty}^{\infty} [(\hat{a}^\dagger)^{n-1} \phi_0]^* \hat{a}\hat{a}^\dagger [(\hat{a}^\dagger)^{n-1} \phi_0] dx \tag{42}$$

Now, $\hat{a}\hat{a}^\dagger$ can be rewritten in the following way:

$$\hat{a}\hat{a}^\dagger = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a} = [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger\hat{a} = 1 + \hat{a}^\dagger\hat{a}, \tag{43}$$

where we have used the identity $[\hat{a}, \hat{a}^\dagger] = 1$ (prove this by expressing $\hat{a}$ and $\hat{a}^\dagger$ in terms of $\hat{p}$ and $\hat{x}$). Inserting this into our expression, we have

$$(\tilde{\phi}_n, \tilde{\phi}_n) = (\tilde{\phi}_{n-1}, \tilde{\phi}_{n-1}) + \int_{-\infty}^{\infty} [(\hat{a}^\dagger)^{n-1} \phi_0]^* \hat{a}^\dagger\hat{a} [(\hat{a}^\dagger)^{n-1} \phi_0] dx \tag{44a}$$

$$= (\tilde{\phi}_{n-1}, \tilde{\phi}_{n-1}) + \int_{-\infty}^{\infty} \tilde{\phi}_{n-1}^* \hat{a}^\dagger\hat{a} \tilde{\phi}_{n-1} dx. \tag{44b}$$

Now, recall that the energy operator for a harmonic oscillator can be written as

$$\hat{E} = \hbar\omega \left(\hat{a}^\dagger\hat{a} + \frac{1}{2} \right), \tag{45}$$

which means we can rewrite the last line in terms of the energy operator:

$$(\tilde{\phi}_n, \tilde{\phi}_n) = (\tilde{\phi}_{n-1}, \tilde{\phi}_{n-1}) - \frac{1}{2}(\tilde{\phi}_{n-1}, \tilde{\phi}_{n-1}) + \frac{1}{\omega} \int_{-\infty}^{\infty} \tilde{\phi}_{n-1}^* \hat{E} \tilde{\phi}_{n-1} dx. \tag{46}$$

This is nice because $\tilde{\phi}_{n-1}$ is an *energy eigenstate* (it may not be normalized, but it's still an energy eigenstate, because a constant times an eigenfunction is still an eigenfunction). This means that $\hat{E}$ acts on it in a very simple way, because it satisfies the energy eigenvalue equation:

$$\hat{H}\tilde{\phi}_{n-1} = E_{n-1}\tilde{\phi}_{n-1} \quad \text{where} \quad E_n = \left(n + \frac{1}{2} \right) \hbar\omega. \tag{47}$$

In our case, then, we have

$$(\tilde{\phi}_n, \tilde{\phi}_n) = \frac{1}{2}(\tilde{\phi}_{n-1}, \tilde{\phi}_{n-1}) + \frac{E_{n-1}}{\omega}(\tilde{\phi}_{n-1}, \tilde{\phi}_{n-1}) \tag{48a}$$

$$= \frac{1}{2}(\tilde{\phi}_{n-1}, \tilde{\phi}_{n-1}) + \left[(n-1) + \frac{1}{2}\right](\tilde{\phi}_{n-1}, \tilde{\phi}_{n-1}) \tag{48b}$$

$$= n(\tilde{\phi}_{n-1}, \tilde{\phi}_{n-1}). \tag{48c}$$

We have thus related the norm square of $\tilde{\phi}_n$ to the norm square of $\tilde{\phi}_{n-1}$. We can repeat the process until we reach the ground state:

$$(\tilde{\phi}_n, \tilde{\phi}_n) = n(\tilde{\phi}_{n-1}, \tilde{\phi}_{n-1}) = n(n-1)(\tilde{\phi}_{n-2}, \tilde{\phi}_{n-2}) = \cdots = n(n-1)\ldots 2 \cdot 1(\tilde{\phi}_0, \tilde{\phi}_0) = n!, \tag{49}$$

where in the last step we used the fact that $\tilde{\phi}_0 = \phi_0$, so $\tilde{\phi}_0$ is in fact the normalized ground state. The norm of $|\tilde{n}\rangle$ is thus

$$\sqrt{(\tilde{\phi}_n, \tilde{\phi}_n)} = \sqrt{n!}\,. \tag{50}$$

(b) **(6 points)** The normalized energy eigenstates are given by

$$\phi_n \equiv \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n \phi_0. \tag{51}$$

First we test for the "normality" of the states:

$$(\phi_n|\phi_n) = \frac{1}{n!}\int_{-\infty}^{\infty} \phi_0^* \hat{a}^n (\hat{a}^\dagger)^n \phi_0 dx = \frac{n!}{n!} = 1, \tag{52}$$

where we made use of the result we proved in (a). For orthogonality, we have

$$(\phi_m|\phi_n) = \frac{1}{\sqrt{m!n!}}\int_{-\infty}^{\infty} \phi_0^* \hat{a}^m (\hat{a}^\dagger)^n \phi_0 dx, \tag{53}$$

where $m \neq n$. Now, after acting on ϕ_0 with the creation operator $\hat{a}^\dagger$ n times, we end up with a state that is proportional to the nth excited state ϕ_n. Our equation then calls for the annihilation operator $\hat{a}$ to act on the state m times. If $m > n$, we have more annihilation operators than we do creation operators, and at some point we get to $\hat{a}\phi_0$, which gives us zero. If $m < n$, we can say

$$\frac{1}{\sqrt{m!n!}}\int_{-\infty}^{\infty} \phi_0^* \hat{a}^m (\hat{a}^\dagger)^n \phi_0 dx = \frac{1}{\sqrt{m!n!}}\int_{-\infty}^{\infty} (\hat{a}^n (\hat{a}^\dagger)^m \phi_0)^* \phi_0 dx, \tag{54}$$

and again we end up lowering a state more than we raise it, only this time it's the left copy of the wavefunction. We thus conclude that $(\phi_m|\phi_n) = 0$ unless $m = n$, in which case $(\phi_m|\phi_n) = 1$. The states ϕ_n are therefore orthonormal.

(c) **(6 points)** We wish to show that

$$\hat{a}\phi_n = \sqrt{n}\phi_{n-1} \quad \text{and} \quad \hat{a}^\dagger\phi_n = \sqrt{n+1}\phi_{n+1}. \tag{55}$$

So far, we only know that $\hat{a}\phi_n$ is *proportional to* ϕ_{n-1}. In other words,

$$\hat{a}\phi_n = D\phi_{n-1}, \tag{56}$$

where D is some proportionality constant. Let us "dot" this equation with itself and perform manipulations similar to what we did in part (a):

$$\begin{aligned}
\int_{-\infty}^{\infty}[\hat{a}\phi_n]^*[\hat{a}\phi_n]dx &= D^2(\phi_{n-1}|\phi_{n-1}) && \text{(57a)}\\
\int_{-\infty}^{\infty}\phi_n^*\hat{a}^\dagger\hat{a}\phi_n dx &= D^2(\phi_{n-1}|\phi_{n-1}) && \text{(57b)}\\
\frac{1}{\hbar\omega}\int_{-\infty}^{\infty}\phi_n^*\hat{E}\phi_n dx - \frac{1}{2}(\phi_n|\phi_n) &= D^2(\phi_{n-1}|\phi_{n-1}) && \text{(57c)}\\
\frac{E_n}{\hbar\omega}(\phi_n|\phi_n) - \frac{1}{2}(\phi_n|\phi_n) &= D^2(\phi_{n-1}|\phi_{n-1}) && \text{(57d)}\\
n(\phi_n|\phi_n) &= D^2(\phi_{n-1}|\phi_{n-1}) && \text{(57e)}\\
\sqrt{n} &= D, && \text{(57f)}
\end{aligned}$$

where we have used the fact that $E_n = \left(n + \frac{1}{2}\right)\ \omega$ and that ϕ_n and ϕ_{n-1} are *normalized* energy eigenstates of the harmonic oscillator. We thus have

$$\hat{a}\phi_n = \sqrt{n}\phi_{n-1}. \tag{58}$$

Similarly, let

$$\hat{a}^\dagger\phi_n = F\phi_{n+1}, \tag{59}$$

where F is some proportionality constant. Performing analogous manipulations, we have:

$$\begin{aligned}
\int_{-\infty}^{\infty}[\hat{a}^\dagger\phi_n]^*[\hat{a}^\dagger\phi_n]dx &= F^2(\phi_{n-1}|\phi_{n-1}) && \text{(60a)}\\
\int_{-\infty}^{\infty}\phi_n^*\hat{a}\hat{a}^\dagger\phi_n dx &= F^2(\phi_{n-1}|\phi_{n-1}) && \text{(60b)}\\
\int_{-\infty}^{\infty}\phi_n^*\ \ [\hat{a},\hat{a}^\dagger] + \hat{a}^\dagger\hat{a}\ \ \phi_n dx &= F^2(\phi_{n-1}|\phi_{n-1}) && \text{(60c)}\\
(\phi_n|\phi_n) + \int_{-\infty}^{\infty}\phi_n^*\hat{a}^\dagger\hat{a}\phi_n dx &= F^2(\phi_{n-1}|\phi_{n-1}) && \text{(60d)}\\
(\phi_n|\phi_n) + n(\phi_n|\phi_n) &= F^2(\phi_{n-1}|\phi_{n-1}) && \text{(60e)}\\
(n+1)(\phi_n|\phi_n) &= F^2 && \text{(60f)}\\
\sqrt{n+1} &= F, && \text{(60g)}
\end{aligned}$$

where we have used the fact that $[\hat{a},\hat{a}^\dagger] = 1$ and the result $D^2 \equiv \int_{-\infty}^{\infty}\phi_n^*\hat{a}^\dagger\hat{a}\phi_n dx = n$ that we obtained above. We thus have

$$\hat{a}^\dagger\phi_n = \sqrt{n+1}\phi_{n+1}. \tag{61}$$

(d) **(6 points)** The number operator is defined as

$$\hat{N} = \hat{a}^\dagger \hat{a}. \tag{62}$$

We now examine the commutators of this operator with the annihilation and creation operators:

$$[\hat{N}, \hat{a}] = \hat{N}\hat{a} - \hat{a}\hat{N} = \hat{a}^\dagger\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{a} = -(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a})\hat{a} = -[\hat{a}, \hat{a}^\dagger]\hat{a} = -\hat{a}. \tag{63}$$

Similarly,

$$[\hat{N}, \hat{a}^\dagger] = \hat{N}\hat{a}^\dagger - \hat{a}^\dagger\hat{N} = \hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^{\hat{a}^\dagger} = \hat{a}^\dagger(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}) = \hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger. \tag{64}$$

From lecture, we know that for the harmonic oscillator the energy operator $\hat{E}$ can be written as[4]

$$\hat{E} = \hbar\omega_0 \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega_0 \left(\hat{N} + \frac{1}{2}\right) \quad \Rightarrow \quad \hat{N} = \frac{\hat{E}}{\hbar\omega_0} - \frac{1}{2}. \tag{65}$$

This means that the eigenfunctions of $\hat{N}$ are the eigenfunctions of $\hat{E}$, *i.e.* the energy eigenstates of the harmonic oscillator. We can see this explicitly by having $\hat{N}$ act on ϕ_n:

$$\hat{N}\phi_n = \frac{1}{\hbar\omega_0}\hat{E}\phi_n - \frac{1}{2}\phi_n = \frac{\hbar\omega_0}{\hbar\omega_0}\left(n + \frac{1}{2}\right)\phi_n - \frac{1}{2}\phi_n = n\phi_n, \tag{66}$$

since we know that the energy eigenvalues of the harmonic oscillator are given by $E_n = \ \omega_0 \ n + \frac{1}{2}$. We can also see from this that the eigenvalues of $\hat{N}$ are simply the various values of n (*i.e.* the quantum number corresponding to each energy eigenstate).

(e) **(6 points)** Our original definitions of $\hat{a}$ and $\hat{a}^\dagger$ were

$$\hat{a} \equiv \frac{\hat{x}}{\sqrt{2\hbar/m\omega_0}} + i\frac{\hat{p}}{\sqrt{2m \ \omega_0}} \quad \text{and} \quad \hat{a}^\dagger \equiv \frac{\hat{x}}{\sqrt{2\hbar/m\omega_0}} - i\frac{\hat{p}}{\sqrt{2m\hbar\omega_0}}. \tag{67}$$

We can solve these to get expression for $\hat{x}$ and $\hat{p}$:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}}(\hat{a} + \hat{a}^\dagger) \quad \text{and} \quad \hat{p} = -i\sqrt{\frac{m\hbar\omega_0}{2}}(\hat{a} - \hat{a}^\dagger). \tag{68}$$

Now, for an eigenstate ϕ_n, we have

$$\langle\hat{x}\rangle = \int_{-\infty}^{\infty} \phi_n^* \hat{x}\phi_n dx = \sqrt{\frac{\hbar}{2m\omega_0}} \int_{-\infty}^{\infty} \phi_n^*(\hat{a} + \hat{a}^\dagger)\phi_n dx = 0, \tag{69}$$

because when $\hat{a}^\dagger$ acts on ϕ_n, the result is proportional to ϕ_{n+1}, which is orthogonal to ϕ_n, and similarly for $\hat{a}$. For the momentum, we have

$$\langle\hat{p}\rangle = \int_{-\infty}^{\infty} \phi_n^* \hat{p}\phi_n dx = -i\sqrt{\frac{m\hbar\omega_0}{2}} \int_{-\infty}^{\infty} \phi_n^*(\hat{a} - \hat{a}^\dagger)\phi_n dx = 0, \tag{70}$$

using exactly the same reasoning.

[4] If you're not sure where this comes from, express $\hat{a}$ and $\hat{a}^\dagger$ in terms of $\hat{x}$ and $\hat{p}$, and make use of the commutation relation $[\hat{x}, \hat{p}] = i\hbar$ to express $\hat{E}$ of the harmonic oscillator in terms of $\hat{a}$ and $\hat{a}^\dagger$.

(f) **(6 points)** First we express $\hat{x}^2$ and $\hat{p}^2$ in terms of $\hat{a}$ and $\hat{a}^\dagger$:

$$\hat{x}^2 = \left[\sqrt{\frac{\hbar}{2m\omega_0}}(\hat{a}+\hat{a}^\dagger)\right]\left[\sqrt{\frac{\hbar}{2m\omega_0}}(\hat{a}+\hat{a}^\dagger)\right] = \frac{\hbar}{2m\omega_0}(\hat{a}+\hat{a}^\dagger)(\hat{a}+\hat{a}^\dagger) \quad (71a)$$

$$= \frac{\hbar}{2m\omega_0}(\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+(\hat{a}^\dagger)^2) = \frac{\hbar}{2m\omega_0}(\hat{a}^2+(\hat{a}^\dagger)^2+2\hat{a}^\dagger\hat{a}+[\hat{a},\hat{a}^\dagger]) \quad (71b)$$

$$= \frac{\hbar}{2m\omega_0}(\hat{a}^2+(\hat{a}^\dagger)^2+2\hat{N}+1). \quad (71c)$$

Similarly, for $\hat{p}^2$, we have

$$\hat{p}^2 = \left[-i\sqrt{\frac{m\hbar\omega_0}{2}}(\hat{a}-\hat{a}^\dagger)\right]\left[-i\sqrt{\frac{m\hbar\omega_0}{2}}(\hat{a}-\hat{a}^\dagger)\right] = -\frac{m\hbar\omega_0}{2}(\hat{a}-\hat{a}^\dagger)(\hat{a}-\hat{a}^\dagger) \quad (72a)$$

$$= -\frac{m\hbar\omega_0}{2}(\hat{a}^2-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+(\hat{a}^\dagger)^2) = -\frac{m\hbar\omega_0}{2}(\hat{a}^2+(\hat{a}^\dagger)^2-2\hat{a}^\dagger\hat{a}-[\hat{a},\hat{a}^\dagger]) \quad (72b)$$

$$= -\frac{m\hbar\omega_0}{2}(\hat{a}^2+(\hat{a}^\dagger)^2-2\hat{N}-1). \quad (72c)$$

Plugging these operators into $\langle\hat{x}^2\rangle$ and $\langle\hat{p}^2\rangle$ gives

$$\langle\hat{x}^2\rangle = \int_{-\infty}^{\infty}\phi_n^*\left[\frac{\hbar}{2m\omega_0}(\hat{a}^2+(\hat{a}^\dagger)^2+2\hat{N}+1)\right]\phi_n dx \quad (73a)$$

$$= \frac{\hbar}{2m\omega_0}\int_{-\infty}^{\infty}\phi_n^*(2\hat{N}+1)\phi_n dx \quad (73b)$$

$$= \frac{\hbar}{2m\omega_0}(2n+1), \quad (73c)$$

where in going from the first line to the second line we used the same reasoning as we did in part (e) *i.e.* orthogonality of ϕ_n and $(\hat{a}^\dagger)^2\phi_n \propto \phi_{n+2}$ or $\hat{a}^2\phi_n \propto \phi_{n-2}$. In going from the second line to the third line we used our results from part (d). The expectation value of the momentum looks similar:

$$\langle\hat{p}^2\rangle = \int_{-\infty}^{\infty}\phi_n^*\left[-\frac{m\hbar\omega_0}{2}(\hat{a}^2+(\hat{a}^\dagger)^2-2\hat{N}-1)\right]\phi_n dx \quad (74a)$$

$$= \frac{m\hbar\omega_0}{2}\int_{-\infty}^{\infty}\phi_n^*(2\hat{N}+1)\phi_n dx \quad (74b)$$

$$= \frac{m\hbar\omega_0}{2}(2n+1). \quad (74c)$$

Now, recall that the uncertainties are defined as $\Delta x \equiv \sqrt{\langle\hat{x}^2\rangle-\langle\hat{x}\rangle^2}$ and $\Delta p \equiv \sqrt{\langle\hat{p}^2\rangle-\langle\hat{p}\rangle^2}$. Since $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ are zero from part (e), we get

$$\Delta x\Delta p = \sqrt{\langle\hat{x}^2\rangle\langle\hat{p}^2\rangle} = \frac{\hbar}{2}(2n+1), \quad (75)$$

which is our desired result.

8.04 Quantum Physics I
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