

Problem 1. (20 points) Mathematical Preliminaries: Linear Operators

(a) **(6 points)** To show that an operator \hat{O} is linear, we need to show that

$$\hat{O}(af(x) + bg(x)) = a\hat{O}f(x) + b\hat{O}g(x). \quad (1)$$

To show that an operator is *not* linear, we simply need to find a counterexample.

- The identity \hat{I} is linear: $\hat{I}(af(x) + bg(x)) = af(x) + bg(x) = a\hat{I}f(x) + b\hat{I}g(x)$.
- Squaring a function is *not* linear. As a counterexample, let $f(x) = x^2$ and $g(x) = x$:

$$\hat{S}(ax^2 + bx) = (ax^2 + bx)^2 = a^2x^4 + 2abx^4 + b^2x^2 \quad (2a)$$

$$a\hat{S}x^2 + b\hat{S}x = ax^4 + bx^2 \quad (2b)$$

So $\hat{S}(af(x) + bg(x)) \neq a\hat{S}f(x) + b\hat{S}g(x)$ and \hat{S} is nonlinear.

- Differentiation is linear:

$$\hat{D}(af(x) + bg(x)) = \frac{\partial}{\partial x}(af(x) + bg(x)) = a\frac{\partial f(x)}{\partial x} + b\frac{\partial g(x)}{\partial x} = a\hat{D}f(x) + b\hat{D}g(x) \quad (3)$$

- Integrating from 0 to x is linear:

$$\begin{aligned} \hat{Q}(af(x) + bg(x)) &= \int_0^x af(x') + bg(x')dx' = a \int_0^x f(x')dx' + b \int_0^x g(x')dx' = \\ &= a\hat{Q}f(x) + b\hat{Q}g(x) \end{aligned} \quad (4)$$

- Adding 3 is, strangely enough, *not* linear. As a counterexample, let $f(x) = x^2$ and $g(x) = x$:

$$\hat{A}(ax^2 + bx) = ax^2 + bx + 3 \quad (5a)$$

$$a\hat{A}x^2 + b\hat{A}x = ax^2 + 3 + bx + 3 = ax^2 + bx + 6 \quad (5b)$$

So $\hat{A}(af(x) + bg(x)) \neq a\hat{A}f(x) + b\hat{A}g(x)$ and \hat{A} is nonlinear.

- Mapping to a fixed function is *not* linear. As a counterexample, let $f(x) = x^2$ and $g(x) = x$:

$$\hat{\mathbb{P}}_h(ax^2 + bx) = h(x) \quad (6a)$$

$$a\hat{\mathbb{P}}_hx^2 + b\hat{\mathbb{P}}_hx = ah(x) + bh(x) \quad (6b)$$

So $\hat{\mathbb{P}}_h(af(x) + bg(x)) \neq a\hat{\mathbb{P}}_hf(x) + b\hat{\mathbb{P}}_hg(x)$ and $\hat{\mathbb{P}}_h$ is nonlinear.

- Translating $f(x)$ by L is linear:

$$\hat{T}_L(af(x) + bg(x)) = af(x - L) + bg(x - L) = a\hat{T}_L f(x) + b\hat{T}_L g(x) \quad (7)$$

- (b) **(8 points)** A *nonvanishing* function $h(x)$ is said to be an eigenfunction of an operator \hat{O} if it satisfies the equation

$$\hat{O}h(x) = \lambda h(x), \quad (8)$$

where λ is some *number* (i.e. *not* a function) called the *eigenvalue*. In other words, an eigenfunction $h(x)$ is some function that happens to have just the right form so that even if the operator \hat{O} is in general very complicated, its action on $h(x)$ is just to multiply the function by some number (which is allowed to be complex).

Note that, if the operator \hat{O} is linear and $h(x)$ is an eigenfunction, also $Ah(x)$, i.e. the function $h(x)$ multiplied by a constant A , is also an eigenfunction of \hat{O} . Indeed,

$$\hat{O}Ah(x) = A\hat{O}h(x) = A\lambda h(x) = \lambda(Ah(x)). \quad (9)$$

Because of this reason, when we deal with linear operators we don't consider the constant A , i.e. we set it equal to 1 and pick only one representative function $h(x)$. Below we shall do the same: for nonlinear operators we consider all the possible eigenfunctions, for linear operators we pick only one eigenfunctions among those which differ only by a proportionality constant.

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$$\hat{1}f(x) = \lambda f(x). \quad (10)$$

Consider first case (i), i.e. the space of arbitrary functions defined on the real line. Since the identity operator $\hat{1}$ maps $f(x)$ to itself, we conclude that any nonvanishing function is an eigenfunction of $\hat{1}$, and that the eigenvalue λ is always 1.

The above discussion holds also for the other three spaces of functions (ii)-(iv).

- Again, let's first consider case (i). To find eigenfunctions and eigenvalues of the square operator \hat{S} , we need to solve the equation

$$f^2(x) = \lambda f(x) \quad (11)$$

where λ is any complex number, and will be the eigenvalue associated to $f(x)$. The solution is

$$f(x) = \lambda, 0 \quad (12)$$

meaning that $f(x)$ can be equal to either λ or 0 at x , for example

$$f(x) = \begin{cases} \lambda & \text{if } x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Note in particular that this function is not continuous, so the discussion so far holds only for case (i). Regarding cases (ii),(iii), $f(x)$ needs to be continuous, and thus the only possibility is that $f(x)$ is a constant equal to λ or 0. Since

an eigenfunction is nonvanishing by definition, all the possible eigenfunctions in cases (ii),(iii) are $f(x) = \lambda$, with $\lambda \neq 0$. Finally, we note that $f(x) = \lambda$ is not square-normalizable, indeed:

$$\int_{-\infty}^{\infty} dx |f(x)|^2 = |\lambda|^2 \int_{-\infty}^{\infty} dx = \infty,$$

therefore when we consider the operator \hat{S} as acting on the space of square-normalizable continuous functions there are no eigenfunctions associated to it.

- The eigenfunction equation for the multiplicative operator \hat{x} is

$$xf(x) = \lambda f(x). \quad (13)$$

In the space of arbitrary functions, the nonvanishing functions which satisfy this equation are $f(x) = \delta(x - x_0)$, for any x_0 , and the corresponding eigenvalues are $\lambda = x_0$. Indeed, from Problem Set 2, we know that the equality $x\delta(x - x_0) = \lambda\delta(x - x_0)$ holds only if

$$\int_a^b x\delta(x - x_0)g(x)dx = \int_a^b \lambda\delta(x - x_0)g(x) \quad (14)$$

for any $g(x)$. We have, for $a < x_0 < b$,

$$\int_a^b x\delta(x - x_0)g(x)dx = x_0g(x_0) = \int_a^b \lambda\delta(x - x_0)g(x) \quad (15)$$

which indeed gives $\lambda = x_0$.

The above discussion holds only for (i). Note that $f(x) = \delta(x - x_0)$ is not a continuous function, and in fact it has an infinite jump at x_0 . Cases (ii)-(iv) require functions to be continuous, so $\delta(x - x_0)$ is excluded for any value of x_0 . This means that in cases (ii)-(iv) there are no eigenfunctions for the operator \hat{x} . One could ask: is the delta function square integrable (although not continuous)? To answer this question, we need to use one of the definitions of delta function that we learnt from Problem Set 2, e.g.

$$\delta(x - x_0) = \lim_{a \rightarrow 0} \frac{1}{a\sqrt{\pi}} e^{-(x-x_0)^2/a^2}.$$

Then we have

$$\begin{aligned} \int_{-\infty}^{\infty} (\delta(x - x_0))^2 &= \lim_{a \rightarrow 0} \int_{-\infty}^{\infty} \left(\frac{1}{a\sqrt{\pi}} e^{-(x-x_0)^2/a^2} \right)^2 = \\ &= \lim_{a \rightarrow 0} \frac{1}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} \frac{\sqrt{2}}{a\sqrt{\pi}} e^{-2(x-x_0)^2/a^2} = \lim_{a \rightarrow 0} \frac{1}{\sqrt{2}a} = \infty, \end{aligned}$$

therefore the delta function is not square-integrable.

- The eigenstate equation for the derivative operator \hat{D} is

$$\hat{D}f(x) = \lambda f(x) \quad (16)$$

i.e.

$$\frac{\partial f(x)}{\partial x} = \lambda f(x) \quad (17)$$

the solution to this equation in the space of arbitrary functions is

$$f(x) = e^{\lambda x}. \quad (18)$$

Thus, the eigenstates of the derivative operator are exponentials, and the eigenvalues are all the complex numbers.

The above discussion applies also to case (ii), because the exponential is a continuous function. However, we should be careful with case (iii), as generally the exponential diverges at infinity. Let's write the complex number $\lambda = a + ib$, where a and b are both real. We have

$$f(x) = e^{\lambda x} = e^{ax} e^{ibx}.$$

If $a > 0$, then $f(x)$ diverges as $x \rightarrow \infty$, and if $a < 0$ then $f(x)$ diverges as $x \rightarrow -\infty$. Consider now $a = 0$. Then $f(x)$ remains finite as $x \rightarrow \pm\infty$, because $|f(x)|^2 = 1$. Thus, in case (iii), the eigenfunctions of \hat{D} are the exponential $e^{\lambda x}$ with $\text{Re}\lambda = 0$, i.e. λ has to be purely imaginary, so that $f(x)$ is non-divergent at infinity. To study case (iv), we need to select the eigenfunctions which are square-normalizable. For any complex $\lambda = a + ib$, we have

$$\int_{-\infty}^{\infty} |f|^2(x) = \int_{-\infty}^{\infty} e^{2ax} = \infty,$$

even when $a = 0$, and thus the exponential can never be square-normalizable. We then conclude that in case (iv) there are no eigenfunctions for \hat{D} .

- (c) **(6 points)** In this case, we require that

$$\phi_{\beta}(x - L) = \hat{T}_L \phi_{\beta}(x) = \alpha \phi_{\beta}(x). \quad (19)$$

Introducing $\phi_{\beta}(x) = e^{\beta x} g(x)$, where $g(x)$ is any function and β is any complex number, the equation becomes

$$e^{\beta(x-L)} g(x - L) = \alpha e^{\beta x} g(x) \quad (20)$$

setting $\alpha = e^{-\beta L}$, the equation simplifies to

$$g(x - L) = g(x) \quad (21)$$

Thus we obtain that the eigenstates of \hat{T}_L are of the form

$$\phi_{\beta}(x) = e^{\beta x} g(x) \quad (22)$$

where $g(x - L) = g(x)$, and the corresponding eigenvalue is $\alpha = e^{-\beta L}$.

Problem 2. (25 points) Mathematical Preliminaries: More on Translation Operator

- (a) **(2 points)** By applying the commutator $[\hat{T}_L, \hat{x}]$ on some arbitrary function $f(x)$ we obtain:

$$[\hat{T}_L, \hat{x}] f(x) = \hat{T}_L(\hat{x}f(x)) - \hat{x}\hat{T}_L f(x) \quad (23)$$

$$= \hat{T}_L(xf(x)) - x\hat{T}_L f(x) \quad (24)$$

$$= (x-L)f(x-L) - xf(x-L) \quad (25)$$

$$= -Lf(x-L) = -L\hat{T}_L f(x). \quad (26)$$

It is important to note that in going from line (19) to line (20) we used the fact that in position space representation the position operator \hat{x} becomes just a multiplicative operator $x \times$.

- (b) **(2 points)** Following the same approach as above we have, acting on some arbitrary function $f(x)$,

$$\hat{T}_L \hat{D} f(x) = \hat{T}_L \frac{\partial}{\partial x} f(x) = \hat{T}_L f'(x) = f'(x-L) \quad (27a)$$

$$\hat{D} \hat{T}_L f(x) = \hat{D} f(x-L) = \frac{\partial}{\partial x} f(x-L) = f'(x-L). \quad (27b)$$

Thus, $\hat{T}_L \hat{D} f(x) - \hat{D} \hat{T}_L f(x) = 0$, and since $f(x)$ was an arbitrary function, $[\hat{T}_L, \hat{D}] = 0$.

- (c) **(4 points)** We have

$$e^{-L \frac{\partial}{\partial x}} = 1 - L \frac{\partial}{\partial x} + \frac{L^2}{2} \frac{\partial^2}{\partial x^2} + \dots \quad (28)$$

If it seems strange to you that “squaring” $\partial/\partial x$ gives the second derivative and not the first derivative squared, I’d suggest thinking about it like this — when thought of as an operator, x^2 means to multiply something by x twice, so the “square” of $\partial/\partial x$ must mean applying the derivative operator twice *i.e.* taking the second derivative. Acting on an arbitrary function $f(x)$ gives

$$e^{-L \frac{\partial}{\partial x}} f(x) = f(x) - L \frac{\partial f}{\partial x} + \frac{L^2}{2} \frac{\partial^2 f}{\partial x^2} + \dots \quad (29)$$

The RHS of this equation looks suspiciously like a Taylor series. A function $f(u)$ can be represented as a Taylor expansion about some other point u_0 a distance $\Delta u \equiv u - u_0$ away:

$$f(u) = f(u_0 + \Delta u) = f(u_0) + \Delta u f'(u_0) + \frac{(\Delta u)^2}{2!} f''(u_0) + \dots \quad (30)$$

Comparing our last two equations, we see that they're really the same thing, with $u_0 \rightarrow x$, $\Delta u \rightarrow -L$. This means that what we have on the RHS of Equation 29 is $f(x - L)$, so

$$e^{-L\frac{\partial}{\partial x}} f(x) = f(x - L) = \hat{T}_L f(x). \quad (31)$$

Since $f(x)$ is arbitrary, it must be generally true that $\hat{T}_L = e^{-L\frac{\partial}{\partial x}}$.

(d) **(4 points)** From part (a) we know that

$$[\hat{T}_L, \hat{x}] = -L\hat{T}_L \quad (32)$$

and from part (c) we know that

$$\hat{T}_L = e^{-L\hat{D}}. \quad (33)$$

Plugging (33) in (32) we obtain

$$[\hat{T}_L, \hat{x}] = -L\hat{T}_L = -L(\hat{\mathbb{1}} - L\hat{D} + \dots) \quad (34)$$

and

$$[\hat{T}_L, \hat{x}] = [\hat{\mathbb{1}} - L\hat{D} + \dots, \hat{x}] = [\hat{\mathbb{1}}, \hat{x}] + [-L\hat{D} + \dots, \hat{x}] = -L[\hat{D} + \dots, \hat{x}] \quad (35)$$

Now, comparing the terms in (34) and (35) with the same powers of L , we obtain in particular, for the term linear in L ,

$$\hat{\mathbb{1}} = [\hat{D}, \hat{x}]. \quad (36)$$

Pushing a bit further this calculation, and writing

$$[\hat{T}_L, \hat{x}] = -L\hat{T}_L = -L \sum_{n=0}^{\infty} \frac{1}{n!} (-L\hat{D})^n = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (-L)^n \hat{D}^{n-1} \quad (37)$$

and

$$[\hat{T}_L, \hat{x}] = \left[\sum_{n=0}^{\infty} \frac{1}{n!} (-L\hat{D})^n, \hat{x} \right] = \sum_{n=0}^{\infty} \frac{1}{n!} (-L)^n [\hat{D}^n, \hat{x}] \quad (38)$$

we obtain, comparing again the terms of the same degree in L between (37) and (38),

$$\frac{1}{(n-1)!} \hat{D}^{n-1} = \frac{1}{n!} [\hat{D}^n, \hat{x}] \quad (39)$$

and thus

$$[\hat{D}^n, \hat{x}] = n\hat{D}^{n-1}. \quad (40)$$

Note that this result tells us that the commutator between \hat{D}^n and \hat{x} can be seen as a derivative with respect to \hat{D} acting on \hat{D}^n .

(e) **(3 points)** We need to find the Fourier transform of $f(x - L)$. We have

$$\begin{aligned} f(x - L) &= \hat{T}_L f(x) = \hat{T}_L \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int dk \hat{T}_L e^{ikx} \tilde{f}(k) = \\ &= \frac{1}{\sqrt{2\pi}} \int dk e^{ik(x-L)} \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} e^{-ikL} \tilde{f}(k) \end{aligned} \quad (41)$$

Thus, the Fourier transform of $f(x - L)$ is $e^{-ikL} \tilde{f}(k)$, as we also worked out in Problem Set 2. The action of \hat{T}_L on $\tilde{f}(k)$ is then

$$\hat{T}_L \tilde{f}(k) = e^{-ikL} \tilde{f}(k) \quad (42)$$

(f) **(3 points)** Let's use the Taylor expansion of \hat{T}_L that we found in part (c):

$$\hat{T}_L \tilde{f}(k) = e^{-L\hat{D}} \tilde{f}(k) = (\hat{1} - L\hat{D} + \dots) \tilde{f}(k) \quad (43)$$

and the Taylor expansion of the result in part (e):

$$\hat{T}_L \tilde{f}(k) = e^{-iLk} \tilde{f}(k) = (1 - ikL + \dots) \tilde{f}(k). \quad (44)$$

By comparing (43) and (44), we obtain, matching the terms linear in L ,

$$\hat{D} \tilde{f}(k) = ik \tilde{f}(k). \quad (45)$$

This result makes sense, indeed we can check it through a direct computation of the action of \hat{D} on $\tilde{f}(k)$:

$$\hat{D} f(x) = \frac{\partial}{\partial x} \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int dk \frac{\partial}{\partial x} e^{ikx} \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} ik \tilde{f}(k) \quad (46)$$

this shows that the Fourier transform of $\hat{D} f(x)$ is $ik \tilde{f}(k)$, which is precisely what we determined in (45).

(g) **(3 points)** We have

$$\begin{aligned} \hat{x} \tilde{f}(k) &= \hat{x} \frac{1}{\sqrt{2\pi}} \int dk e^{-ikx} f(x) = \frac{1}{\sqrt{2\pi}} \int dk \hat{x} e^{-ikx} f(x) = \\ &= \frac{1}{\sqrt{2\pi}} \int dk i \frac{\partial}{\partial k} e^{-ikx} f(x) = i \frac{\partial}{\partial k} \frac{1}{\sqrt{2\pi}} \int dk e^{-ikx} \tilde{f}(x) \end{aligned} \quad (47)$$

This tells us that

$$\hat{x} \tilde{f}(k) = i \frac{\partial}{\partial k} \tilde{f}(k) \quad (48)$$

(h) **(4 points)** We have, for $f(x)$,

$$[\hat{D}, \hat{x}] f(x) = \frac{\partial}{\partial x} (x f(x)) - x \frac{\partial}{\partial x} f(x) = f(x) \quad (49)$$

and for $\tilde{f}(k)$,

$$[\hat{D}, \hat{x}] \tilde{f}(k) = iki \frac{\partial}{\partial k} \tilde{f}(k) - i \frac{\partial}{\partial k} (ik \tilde{f}(k)) = \tilde{f}(k) \quad (50)$$

This shows that the commutation relation between \hat{D} and \hat{x} holds for both $f(x)$ and $\tilde{f}(k)$. The lesson we take is that an operator statement does not depend on the choice of representation (either position or momentum) of the functions we consider.

Problem 3. (25 points) Mathematical Preliminaries: The Meaning of the Commutator

- (a) **(6 points)** To show that \hat{C} is a linear operator, we need to apply the definition of linear operator to the commutator. Consider the usual linear combination of two functions $af(x) + bg(x)$. Then we have

$$\hat{A}\hat{B}(af(x) + bg(x)) = \hat{A}(a\hat{B}f(x) + b\hat{B}g(x)) = a\hat{A}\hat{B}f(x) + b\hat{A}\hat{B}g(x) \quad (51)$$

$$\hat{B}\hat{A}(af(x) + bg(x)) = \hat{B}(a\hat{A}f(x) + b\hat{A}g(x)) = a\hat{B}\hat{A}f(x) + b\hat{B}\hat{A}g(x) \quad (52)$$

thus

$$\begin{aligned} [\hat{A}, \hat{B}] &= (\hat{A}\hat{B} - \hat{B}\hat{A})(af(x) + bg(x)) = \hat{A}\hat{B}(af(x) + bg(x)) - \hat{B}\hat{A}(af(x) + bg(x)) = \\ &= a\hat{A}\hat{B}f(x) + b\hat{A}\hat{B}g(x) - a\hat{B}\hat{A}f(x) + b\hat{B}\hat{A}g(x) = a[\hat{A}, \hat{B}]f(x) + b[\hat{A}, \hat{B}]g(x). \end{aligned} \quad (53)$$

- (b) **(7 points)** From the following relations

$$\hat{A}\hat{B}\phi_{ab} = \hat{A}b\phi_{ab} = ab\phi_{ab} \quad (54)$$

$$\hat{B}\hat{A}\phi_{ab} = \hat{B}a\phi_{ab} = ba\phi_{ab} \quad (55)$$

we infer that the action of \hat{A} and \hat{B} on ϕ_{ab} can be interchanged, precisely because the eigenvalues a and b are complex numbers, and so they commute. We conclude that

$$[\hat{A}, \hat{B}]\phi_{ab} = (ab - ba)\phi_{ab} = 0 \quad (56)$$

- (c) **(5 points)** Since \hat{A} and \hat{B} commute, they can share common eigenfunctions.
- (d) **(7 points)** It \hat{A} and \hat{B} are two commuting operators, it's not necessary that an eigenfunction of the former is also an eigenfunction of the latter. As a simple counterexample, take $\hat{A} = \hat{1}$, $\hat{B} = \hat{x}$, and the function $f(x) = x^2$. Evidently, $[\hat{A}, \hat{B}] = 0$, and x^2 is an eigenfunction of \hat{A} with eigenvalue 1. Nevertheless, as we have seen in Problem 1, x^2 is not an eigenfunction of \hat{B} .

Problem 4. (30 points) Operators and Commutators in Quantum Mechanics

- (a) **(3 points)** We have, similarly to Problem 1,

$$[\hat{p}, \hat{x}]\psi(x) = \frac{\hbar}{i}\partial_x(x\psi(x)) - x\frac{\hbar}{i}\partial_x\psi(x) = \frac{\hbar}{i}\psi(x) \quad (57)$$

- (b) **(4 points)** Since the commutator of \hat{x} and \hat{p} is nonvanishing on any function, they can't have any common eigenfunction. Indeed, suppose that $\psi(x)$ is an eigenfunction for both \hat{x} and \hat{p} with eigenvalues a_x, a_p , respectively. Then

$$\frac{\hbar}{i}\psi(x) = [\hat{x}, \hat{p}]\psi(x) = (a_x a_p - a_p a_x)\psi(x) = 0 \quad (58)$$

which gives a contradiction.

- (c) **(4 points)** The operators \hat{p} and \hat{T}_L do share common eigenfunctions. Consider $\psi(x) = e^{ikx}$. Then

$$\hat{p}\psi(x) = \hbar k\psi(x) \quad (59)$$

$$\hat{T}_L\psi(x) = e^{-ikL}\psi(x). \quad (60)$$

Physically, this tells us that states with definite momentum are translationally invariant up to an overall phase. We will return to this example later in the semester, where it will play an important role in explaining the physics of solids.

- (d) **(3 points)** In order for \hat{A} and \hat{B} to share an eigenfunction ϕ , we need that $[\hat{A}, \hat{B}]\phi = 0$. ϕ is thus an eigenfunction of the commutator with eigenvalue zero. Thus, in order to share an eigenfunction, the commutator must have at least one zero eigenvalue. For example, this does not happen for \hat{x} and \hat{p} , which commute to the identity. As we argued in part (b), all eigenvalues of the identity are non-zero, and thus \hat{x} and \hat{p} can share no common eigenfunctions.
- (e) **(3 points)** According to classical mechanics, given sufficiently precise measurements, all observables can in principle be determined with total certainty. However, as we saw with the boxes, or the 2-slit experiment, this is empirically false: in some situations, knowledge of one observable can imply irreducible uncertainty about other observables. In quantum mechanics, this remarkable fact is encoded by representing observables with operators and the state as a wavefunction. A quantum observable can thus only be said to have a well-defined value when the state (the wavefunction) is an eigenfunction of the corresponding operator. ¹

¹Conversely, if you pick a random state for the system, your operator will in general not have any well-defined value, though the probability for measuring any particular value can be determined.

When can two observables have definite values simultaneously? To have such certainly, the wavefunction must be a simultaneous eigenfunction of both of the corresponding operators. But such a shared eigenfunction can only exist if the commutator of the two operators can vanish. Thus, if the commutator of two observables does not vanish, there is an irreducible uncertainty in the values of those observables.

- (f) **(3 points)** \hat{H} and \hat{C} do not commute. The reason for this is that an electron with a definite color can't have a definite hardness. Indeed, recall from Lecture 1 that if we throw a beam of white electrons through a hardness box, the outcome will be 50% white and 50% black. If \hat{H} and \hat{C} were commuting observables, we would be able to select *simultaneously* a definite value of hardness and a definite value of color. To be more explicit, consider the experiment from Lecture 1 where we throw a beam of electrons through the Color box, the Hardness box, and then the Color box again. More precisely, what we do is to:

- measure color, select white electrons
- measure hardness, select soft electrons
- measure color again.

We saw that the final outcome is 50% white electrons and 50% black electrons, instead of being 100% white electrons. This means that measuring hardness interferes with measuring color, or in other words, it is impossible to know hardness and color at the same time. Thus, the uncertainty between hardness and color is nonzero, and we conclude that the hardness and color observables have nonvanishing commutator:

$$[\hat{H}, \hat{C}] \neq 0. \quad (61)$$

- (g) **(6 points)** We need to find a representation of the four functions which gives the eigenvalues required by the problem, and such that the commutator between \hat{H}, \hat{C} does not vanish. ϕ_W has to be a superposition of ϕ_H and ϕ_S , i.e.,

$$\phi_W = \alpha\phi_H + \beta\phi_S. \quad (62)$$

Since a white electron has equal probability of being hard or soft, we impose $|\alpha|^2 = |\beta|^2 = 1/2$. The most general possible superposition can then be written as

$$\phi_W = \left(\frac{1}{\sqrt{2}}\phi_H + \frac{1}{\sqrt{2}}e^{i\theta}\phi_S \right) e^{i\phi}. \quad (63)$$

As argued in class, the overall phase is immaterial, and we can set $\phi = 0$, which leaves us with

$$\phi_W = \frac{1}{\sqrt{2}}\phi_H + \frac{1}{\sqrt{2}}e^{i\theta}\phi_S. \quad (64)$$

A simple guess for ϕ_W could then be

$$\phi_W = \frac{1}{\sqrt{2}}(\phi_H + \phi_S). \quad (65)$$

We also want ϕ_B to have the same properties of ϕ_W , but of course it has to be a different superposition. The simplest combination we can try is

$$\phi_B = \frac{1}{\sqrt{2}}(\phi_H - \phi_S). \quad (66)$$

and verify that these states satisfy the requirements we need. The operators \hat{H}, \hat{C} are already defined in terms of the eigenvalues associated to the four functions. Given this representation, the hardness of ϕ_w is given by the action of \hat{H} on it:

$$\hat{H}\phi_W = \hat{H} \frac{1}{\sqrt{2}}(\phi_H + \phi_S) = \frac{1}{\sqrt{2}}(\phi_H - \phi_S) = \phi_B \quad (67)$$

this equation tells us that ϕ_W does not have a definite hardness, because it's not an eigenfunction of \hat{H} . Analogously,

$$\hat{H}\phi_B = \hat{H} \frac{1}{\sqrt{2}}(\phi_H - \phi_S) = \frac{1}{\sqrt{2}}(\phi_H + \phi_S) = \phi_W \quad (68)$$

so, neither ϕ_B has a definite hardness. Knowing that

$$\phi_H = \frac{1}{\sqrt{2}}(\phi_W + \phi_B), \quad \phi_S = \frac{1}{\sqrt{2}}(\phi_W - \phi_B) \quad (69)$$

We can also check the color of hard and soft electrons:

$$\hat{C}\phi_H = \hat{C} \frac{1}{\sqrt{2}}(\phi_W + \phi_B) = \frac{1}{\sqrt{2}}(-\phi_W + \phi_B) = -\phi_S \quad (70)$$

$$\hat{C}\phi_S = \hat{C} \frac{1}{\sqrt{2}}(\phi_W - \phi_B) = \frac{1}{\sqrt{2}}(-\phi_W - \phi_B) = -\phi_H \quad (71)$$

thus, hard electrons do not have definite colors, as well as black electrons. Last but not least, let's also check the commutator between \hat{H} and \hat{C} :

$$[\hat{H}, \hat{C}]\phi_H = -\hat{H}\phi_S - \hat{C}\phi_H = \phi_S + \phi_S = 2\phi_S \quad (72)$$

$$[\hat{H}, \hat{C}]\phi_S = -\hat{H}\phi_H + \hat{C}\phi_S = -\phi_H - \phi_H = -2\phi_H \quad (73)$$

$$[\hat{H}, \hat{C}]\phi_W = -\hat{H}\phi_W - \hat{C}\phi_B = -\phi_B - \phi_B = -2\phi_B \quad (74)$$

$$[\hat{H}, \hat{C}]\phi_B = \hat{H}\phi_B - \hat{C}\phi_W = \phi_W + \phi_W = 2\phi_W \quad (75)$$

As anticipated in the part (f), the commutator is nonvanishing. The fact that it's nonvanishing on any eigenfunction, means that there exist no function with a precise value for both hardness and color.

- (h) **(4 points)** It is difficult to construct a theory based on nonlinear operators that is consistent with the principle of superposition and the probability interpretation. Recall that the principle of superposition says that if $\psi_1(x)$ and $\psi_2(x)$ are suitable wavefunctions for describing a particle in a given system, then $\psi(x) \equiv \psi_1(x) + \psi_2(x)$ is also an acceptable solution. Now, let's consider the "map-to-g(x)" operator $\hat{\mathbb{P}}_g$ from

Problem 1 and consider what happens when this operator acts on ψ_1 , ψ_2 , and ψ . The first two wavefunctions give us no trouble:

$$\hat{\mathbb{P}}_g \psi_1 = g(x) \tag{76a}$$

$$\hat{\mathbb{P}}_g \psi_2 = g(x) \tag{76b}$$

But what about $\hat{\mathbb{P}}_g \psi$? One answer would be

$$\hat{\mathbb{P}}_g \psi = g(x). \tag{77}$$

On the other hand, we could argue that the answer ought to be

$$\hat{\mathbb{P}}_g \psi_1 + \hat{\mathbb{P}}_g \psi_2 = g(x) + g(x) = 2g(x), \tag{78}$$

which is different because $\hat{\mathbb{P}}_g$ is nonlinear. Physically, though, there can only be one answer, and it shouldn't make a difference how we worked it out — how does a wavefunction “know” whether it's secretly the superposition of two other wavefunctions or not? It doesn't. A wavefunction is a wavefunction.

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