Problem Set 1 **Solutions**
8.04 Spring 2013 **February 13, 2013**

Problem 1. (15 points) Radiative collapse of a classical atom

(a) **(5 points)** We begin by *assuming* that the orbit is circular. This seems like circular[1] logic, but is actually a fairly common technique in physics — what we're trying to do here is to see if our assumptions are self-consistent. As we work through the problem, the math will tell us if we're wrong and need to revise our starting point.

The centripetal force keeping the electron in orbit around the proton is provided by Coulomb force. Thus,

$$\frac{q^2}{r^2} = \frac{mv^2}{r} \quad \Rightarrow \quad v = \sqrt{\frac{q^2}{mr}}, \tag{1}$$

where we have used c.g.s. units. The kinetic energy is given by the usual Newtonian formula:

$$K = \frac{1}{2}mv^2 = \frac{q^2}{2r}. \tag{2}$$

The problem asks us to compare this to the energy lost *per orbit*. The Larmor formula, on the other hand, gives us the energy lost *per time*, so we need to figure out the orbital period of the electron. Since we have assumed that the orbits are circular, we can also assume that the angular speed of the electron is constant. This means the time taken for one orbit is given by

$$\tau = \frac{2\pi r}{v}. \tag{3}$$

The change in energy per orbit is therefore

$$\Delta E \approx \frac{dE}{dt}\Delta t = -\frac{2}{3}\frac{q^2 a^2}{c^3}\tau = -\frac{2}{3}\frac{q^2}{c^3}\left(\frac{v^2}{r}\right)^2\left(\frac{2\pi r}{v}\right) = -\frac{4\pi}{3}\frac{q^2}{r}\left(\frac{v}{c}\right)^3, \tag{4}$$

where we have used the fact that the acceleration is centripetal, so $a = v^2/r$. The ratio of the two energies is

$$\frac{|\Delta E|}{K} = \frac{8\pi}{3}\left(\frac{v}{c}\right)^3 \ll 1, \tag{5}$$

where the last inequality is true as long as the electron is non-relativistic (*i.e.* as long as $v \ll c$), which we shall see in part c is indeed the case for our system. The energy lost is per orbit is thus negligible compared to the kinetic energy of the electron, and we can safely assume that the orbits are circular at any given instant.

(b) **(5 points)** The Larmor formula gives the change in *energy* per time, whereas we're interested in the change in *radius*. We therefore need to relate the energy to the radius:

[1] Pun, admittedly, intended.

$$E = K + U = \frac{1}{2}mv^2 - \frac{q^2}{r} \tag{6a}$$

$$= \frac{q^2}{2r} - \frac{q^2}{r} = -\frac{q^2}{2r}. \tag{6b}$$

The fact that the total energy turned out to be such a simple function of radius after we inserted Equation 1 into the expression is not an accident — this turns out to be a simple case of what's known as the *virial theorem*. Substituting this into the Larmor formula gives

$$\frac{dE}{dt} = \frac{d}{dt}\left(-\frac{q^2}{2r}\right) = -\frac{2}{3}\frac{q^2a^2}{c^3} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{1}{r}\right) = \frac{4}{3c^3}\left(\frac{v^2}{r}\right)^2, \tag{7}$$

where we have once again used the fact that $a = v^2/r$. Once again, we can substitute Equation 1 into this, and after a little algebra one gets

$$\frac{d}{dt}\left(\frac{1}{r}\right) = \frac{4}{3}\frac{q^4}{m^2c^3r^4}. \tag{8}$$

We now proceed to solve the differential equation:

$$-\frac{1}{r^2}\frac{dr}{dt} = \frac{4}{3}\frac{q^4}{m^2c^3r^4} \tag{9a}$$

$$r^2\frac{dr}{dt} = -\frac{4}{3}\frac{q^4}{m^2c^3} \tag{9b}$$

$$\int_{r_i}^{r_f} r^2 dr = -\frac{4}{3}\frac{q^4}{m^2c^3}\int_0^{t_0} dt \tag{9c}$$

$$\frac{1}{3}\left(r_f^3 - r_i^3\right) = -\frac{4}{3}\frac{q^4}{m^2c^3}t_0 \tag{9d}$$

$$\Rightarrow t_0 = \frac{m^2c^3}{4q^4}\left(r_f^3 - r_i^3\right). \tag{9e}$$

Plugging in $r_i = 1\,\text{Å}$ and $r_f = 1\,\text{fm}$ gives a lifetime t_0 of 1.1×10^{-10} seconds. Given that most of us have been around for more than that amount of time, this does not bode well for the classical model of the atom!

(c) **(3 points)** From before, we have $v = \sqrt{\frac{q^2}{mr}}$. Plugging the initial value $r = 0.5\,\text{Å}$ into this gives $v = 2.3 \times 10^8$ cm/s. This is fairly small compared to the speed of light:

$$\frac{v}{c} = \frac{2.3 \times 10^8\ \text{cm/s}}{3 \times 10^{10}\ \text{cm/s}} = 0.0075 \approx 1\%. \tag{10}$$

Since leading order relativistic corrections tend to be of order v^2/c^2, we can expect our non-relativistic analysis to be accurate to one part in 10^4. This calculation also justifies the assumption we made in part (a). Note that the non-relativistic approximation breaks down before the electron reaches the proton. Indeed, assuming that the electron remains non-relativistic all the way to r_f, we would have

$$v_f = \sqrt{\frac{q^2}{mr_f}} = 7.3 \times 10^{10} \text{ cm/s} \qquad \Rightarrow \qquad \frac{v_f}{c} = 240\% \text{ !!}$$

We can consider the non-relativistic approximation to be valid until the orbit becomes about 100 times smaller than the initial value 0.5 Å, where the velocity becomes 10 times bigger, and

$$\frac{v}{c} \approx 10\%$$

so that relativistic corrections are still small, being one part in 10^2.
Those who are interested in how the leading order relativistic correction to this problem can be computed are encouraged to take a look at

www.physics.princeton.edu/∼mcdonald/examples/orbitdecay.pdf

The relativistic analysis requires not only a correction to the dynamical equations, but also to the Larmor formula.

(d) **(2 points)** As the electron approaches the proton ($r \to 0$), the energy approaches $-\infty$, as we can see from Equation 6b. There is no minimum energy in this classical model, which is in contrast to the "real" quantum mechanical atom where there is a well-defined ground state.

Problem 2. (25 points) Dimensional Analysis: Two Kinds of Quantum Gravity

(a) Gravitational bound states

i. **(3 points)** In S.I. units, mass is measured in kg, gravitational acceleration in m/s^2, and Planck's constant in J·s, which is equivalent to $\mathrm{kg\,m^2 s^{-1}}$. We wish to find the combination of mass, gravitational acceleration, and Planck's constant that has units of energy. Writing $E \sim m^\alpha g^\beta \hbar^\gamma$, we have the following units:

$$\mathrm{kg\,m^2\,s^{-2}} = \mathrm{kg}^\alpha (\mathrm{m\,s^{-2}})^\beta (\mathrm{kg\,m^2\,s^{-1}})^\gamma = \mathrm{kg}^{\alpha+\gamma}\mathrm{m}^{\beta+2\gamma}\mathrm{s}^{-2\beta-\gamma}. \tag{11}$$

Equating the exponents gives

$$1 = \alpha + \gamma \tag{12a}$$
$$2 = \beta + 2\gamma \tag{12b}$$
$$-2 = -2\beta - \gamma. \tag{12c}$$

With three equations and three unknowns, we can solve for each exponent, giving $\alpha = 1/3$, $\beta = 2/3$, and $\gamma = 2/3$. Thus,

$$E \sim \left(mg^2\hbar^2\right)^{\frac{1}{3}}, \tag{13}$$

by dimensional analysis. A characteristic energy cannot be found if one does not use Planck's constant, because this would be equivalent to setting $\gamma = 0$, and the resulting system of equations would have no solution.

ii. **(9 points)** To get a characteristic length, time, and speed for this system, we can follow the same procedure as in the previous part, equating the exponents on the RHS of Equation 11 with appropriate values. This yields

$$l \sim \left(\frac{\hbar^2}{m^2 g}\right)^{\frac{1}{3}} \tag{14a}$$
$$t \sim \left(\frac{\hbar}{m g^2}\right)^{\frac{1}{3}} \tag{14b}$$
$$v \sim \left(\frac{g\hbar}{m}\right)^{\frac{1}{3}}. \tag{14c}$$

iii. **(2 points)** Having the particle sit completely still would mean that there would be no uncertainty in its momentum (*i.e.* $\Delta p = 0$), and having it be precisely on the surface would mean no uncertainty in its position (*i.e.* $\Delta x = 0$). Having both Δp and Δx be zero is a clear violation of Heisenberg's Uncertainty Principle, which says $\Delta x \Delta p \geq \hbar/2$.

iv. **(3 points)** The energy of the system is given by $E = K + U = \frac{1}{2}mv^2 + mgx$. Plugging in the characteristic length l found in part (i) for x and the characteristic velocity found in part (ii) for v, we have

$$E_{g.s.} = \frac{1}{2}mv^2 + mgx \sim \frac{1}{2}m\left(\frac{g\hbar}{m}\right)^{\frac{2}{3}} + mg\left(\frac{\hbar^2}{m^2 g}\right)^{\frac{1}{3}} = \frac{3}{2}\left(mg^2\hbar^2\right)^{\frac{1}{3}}, \tag{15}$$

where the 3/2 prefactor can't *really* be trusted, since dimensional estimates like this one can only be expected to be accurate to an order-of-magnitude. Note that it's not entirely obvious that this energy is the *ground state* energy of the system. In other words, while our expression is definitely *some* characteristic energy of the system, it's not clear that it's the *lowest* energy. To see why our dimensional estimate represents the ground state, consider the following alternate derivation. If the system is in its lowest energy state, then we expect the position and momentum of the particle to be as low as possible without violating the uncertainty principle. Thus, we expect

$$x \cdot p \sim \hbar. \tag{16}$$

Since $p = mv$, we can substitute this into our expression for the energy:

$$E = \frac{1}{2}mv^2 + mg\left(\frac{\hbar}{p}\right) = \frac{1}{2}mv^2 + \frac{g\hbar}{v}. \tag{17}$$

The ground state is by definition the *minimum* energy state, so we differentiate with respect to velocity, our only remaining free parameter:

$$\frac{dE}{dv} = mv - \frac{g\hbar}{v^2} = 0 \quad \Rightarrow \quad v = \left(\frac{g\hbar}{m}\right)^{\frac{1}{3}}, \tag{18}$$

which is the characteristic velocity we found before. Plugging this into our expression for the energy gives the same answer as we got in Equation 15.

As $\hbar \to 0$, the predicted ground state energy also approaches zero. This is a mathematical statement of the fact that in classical mechanics, the lowest energy state of the system is one where the particle is sitting at rest on the surface.

v. **(1 point)** Plugging in the mass of a neutron ($m \approx 1.7 \times 10^{-27}$ kg) gives $E \approx 1.21 \times 10^{-31}$ J, $l \approx 7.3 \times 10^{-6}$ m, $t \approx 8.6 \times 10^{-4}$ s, and $v \approx 8.5 \times 10^{-3}$ m/s. We thus expect the particle to be typically be on the order of 10 μm above the surface.

(b) The Planck Scale

i. **(3 points)** The S.I. units for G_N are $\mathrm{kg^{-1}m^3s^{-2}}$, the units for $\hbar$ are $\mathrm{kg\,m^2\,s^{-1}}$, and the units for c are $\mathrm{m\,s^{-1}}$. Repeating the same sort of analysis as we did for part (a), we have

$$\mathrm{m} = \left(\mathrm{kg^{-1}\,m^3\,s^{-2}}\right)^{\alpha}(\mathrm{m\,s^{-1}})^{\beta}(\mathrm{kg\,m^2\,s^{-1}})^{\gamma} = \mathrm{kg}^{-\alpha+\gamma}\mathrm{m}^{3\alpha+\beta+2\gamma}\mathrm{s}^{-2\alpha-\beta-\gamma} \tag{19}$$

$$
\begin{aligned}
0 &= -\alpha + \gamma && (20a) \\
1 &= 3\alpha + \beta + 2\gamma && (20b) \\
0 &= -2\alpha - \beta - \gamma && (20c)
\end{aligned}
$$

This gives $\alpha = 1/2$, $\beta = -3/2$, and $\gamma = 1/2$, so the Planck length is given by

$$L_P \equiv \sqrt{\frac{G\hbar}{c^3}}. \tag{21}$$

ii. **(1 point)** Inserting $\hbar = 1.05 \times 10^{-34}\,\mathrm{J\,s}$, $G = 6.67 \times 10^{-11}\mathrm{kg}^{-1}\,\mathrm{m}^3\,\mathrm{s}^{-2}$, and $c = 3.00 \times 10^8\,\mathrm{m/s}$ into Equation 21 gives $L_P = 1.62 \times 10^{-35}\,\mathrm{m}$. This is roughly 20 orders of magnitude smaller than typical nuclear scales ($\sim 1\,\mathrm{fm}$).

iii. **(3 points)** Following the same procedure gives

$$M_P \equiv \sqrt{\frac{\hbar c}{G}} = 2.18 \times 10^{-8}\,\mathrm{kg}. \tag{22}$$

This is about 19 orders of magnitude larger than the proton mass ($\approx 1.67 \times 10^{-27}\,\mathrm{kg}$). Given this and the answer to part (ii), we see that there is no need for a quantum theory of gravity if one's goal is to study nuclear physics.

Problem 3. (20 points) deBroglie Relations and the Scale of Quantum Effects

(a) Light Waves as Particles

i. **(3 points)** To compute the frequency, we use the formula $c = \lambda\nu$. Plugging in $400 - 700\,\text{nm}$ gives a frequency range of $4.28 \times 10^{14}\,\text{Hz}$ to $7.49 \times 10^{14}\,\text{Hz}$. The photon energy is computed using $E = h\nu$, giving $2.84 \times 10^{-19}\,\text{J} = 1.77\,\text{eV}$ to $4.97 \times 10^{-19}\,\text{J} = 3.10\,\text{eV}$.

ii. **(2 points)** At 2.5 GHz, each photon carries an energy of $E = h\nu = 1.66 \times 10^{-17}$ ergs. If the microwave oven operates at a power of 7.5×10^9 erg/s, it must emit $7.5 \times 10^9 / 1.66 \times 10^{-17} = 4.53 \times 10^{26}$ photons per second. Performing the same calculations for the laser gives 3.19×10^{15} photons per second, and 7.10×10^{23} photons per second for the cell phone.

iii. **(3 points)**A 200 ml glass of water has a mass of 0.2 kg, and to find the amount of energy needed to raise its temperature by 10 K, we can use the formula $Q = mc_V\Delta T$:

$$Q = mc_V\Delta T = (200\,\text{g})\left(4.18 \times 10^7\,\frac{\text{erg}}{\text{g K}}\right)(10\,\text{K}) = 8.36 \times 10^{10}\,\text{ergs}. \tag{23}$$

Dividing by the energy per photon in the previous part tells us that 5.04×10^{27} photons are required to heat up the water.

iv. **(2 points)** X-rays are at a much high frequency than radio waves are, which means (according to $E = h\nu$) that an X-ray photon carries much more energy than a radio photon. Thus, an electromagnetic wave of a given power contains many more photons at radio frequencies than it would at X-ray frequencies. The radio wave is therefore more amenable to being described by a *statistical* description of photons (*i.e.* a *classical* description). This is why radio astronomers almost never have to use quantum mechanics, whereas X-ray astronomers use a photon description of light in their work. (However, see http://arxiv.org/abs/0801.0441 for an interesting twist on this in the context of some recent radio astronomy research).

(b) Matter Particles as Waves

i. **(2 points)** $\lambda = h/p = 1.48 \times 10^{-38}\,\text{m}$.

ii. **(2 points)** $\lambda = h/p = 6.63 \times 10^{-31}\,\text{m}$.

iii. **(2 points)** Model the smoke particle as a sphere. The mass of one such particle is thus $m = \frac{4}{3}\pi\left(\frac{d}{2}\right)^3\rho = 1.05 \times 10^{-16}$ g. With this, we can calculate the speed of a typical molecule using $\frac{1}{2}mv^2 = \frac{3}{2}k_BT$. This gives $v = 0.344\,\text{m/s}$, which when substituted into the de Broglie formula gives $\lambda = 1.84 \times 10^{-14}\,\text{m}$.

iv. **(2 points)** Repeating the same procedure as above yields 27.0 nm.

(2 points) For most everyday objects, we do not notice their wave-like behavior because their wavelengths are far too small to be noticeable. Only with the example in part (iv) do we get to the point where quantum effects might be measurable.

Problem 4. (15 points) Double-slit interference of electrons

(a) **(5 points)** Recall the condition for constructive interference in double slit interference:

$$d \sin \theta_m = m\lambda, \tag{24}$$

where m is the order of the maximum. Calling y the coordinate on the screen where the pattern is projected, we have

$$\sin \theta_m = \frac{y_m}{D}, \tag{25}$$

where y_m is the position of the maximum of order m. Thus

$$\frac{d}{D} y_m = m\lambda. \tag{26}$$

Since $w = y_{m+1} - y_m$, we have

$$\frac{d}{D} w = \lambda \quad \Rightarrow \quad w = \lambda \frac{D}{d}. \tag{27}$$

Using the de Broglie relation $\lambda = \frac{h}{p}$, we obtain

$$w = \frac{hD}{pd}. \tag{28}$$

(b) **(7 points)** To calculate the electron's de Broglie wavelength, we need to use the de Broglie relation $\lambda = h/p$, and we must know the momenta of the electrons. The electrons begin at rest and are accelerated by a voltage of 50 kV, acquiring some kinetic energy:

$$\frac{p^2}{2m} = q\Delta V \quad \Rightarrow \quad p = \sqrt{2mq\Delta V}. \tag{29}$$

This gives a de Broglie wavelength of

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2qm\Delta V}}. \tag{30}$$

Now, $d = 2 \times 10^{-4}$ cm, $D = 35$ cm, $\Delta V = 50$ kV, $h = 6.626 \times 10^{-34}$ m^{2}Kg/s, $e = 1.6 \times 10^{-19}$ C, $m = 9.1 \times 10^{-31}$ kg, thus

$$\lambda = \frac{6.626 \times 10^{-34}}{\sqrt{2 \times 1.6 \times 10^{-19} \times 9.1 \times 10^{-31} \times 50 \times 10^3}} = 5.49 \times 10^{-12} \text{ m},$$

and

$$w = \lambda \frac{D}{d} = 5.49 \times 10^{-12} \frac{35}{2 \times 10^{-4}} = 9.6 \times 10^{-7} \text{ m}.$$

(c) **(3 points)** Visible light ranges roughly between 400 and 700 nm. Let's choose $\bar{\lambda} = 550$ nm. The proportionality factor between this wavelength and the de Broglie wavelength of the electron in the Jönsson experiment is

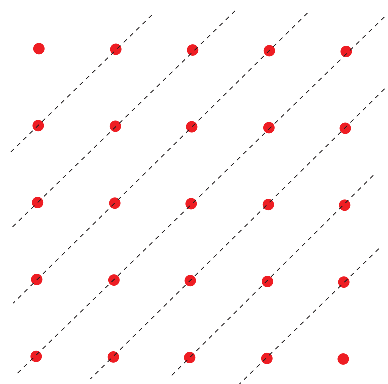
$$a = \frac{\bar{\lambda}}{\lambda} = \frac{550}{5.49 \times 10^{-3}} = 100182.$$

The suitable values for d, D, w are then $\bar{d} = ad = 20$ cm, $\bar{D} = aD = 35064$ m, $\bar{w} = aw = 9.6$ cm.

Problem 5. (15 points) Electron Diffraction

(a) **(2 points)** (Yep, just for watching the video!)

(b) **(3 points)** With diffraction slits in both the horizontal and vertical directions, we can consider the grating as a square grid of emitting sources. In the figure below, one can see that these sources can be grouped into diagonal lines of sources that interfere with each other to give diagonally aligned interference patterns.



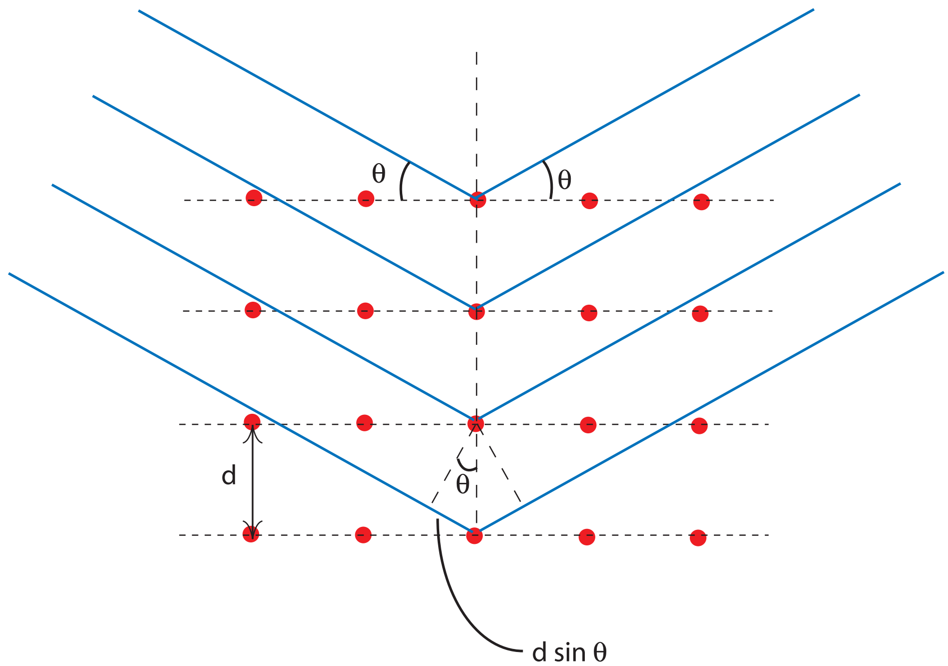
(c) **(10 points)** As obtained in the previous problem, the de Broglie wavelength of electrons accelerated by a potential ΔV is

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{\sqrt{2qm\Delta V}}. \tag{31}$$

The strong reflection of electrons at $50°$ is a result of the constructive interference of electron matter waves (which have the wavelength we just computed). The Bragg relation describes the condition under which waves scatter off planes of atoms:

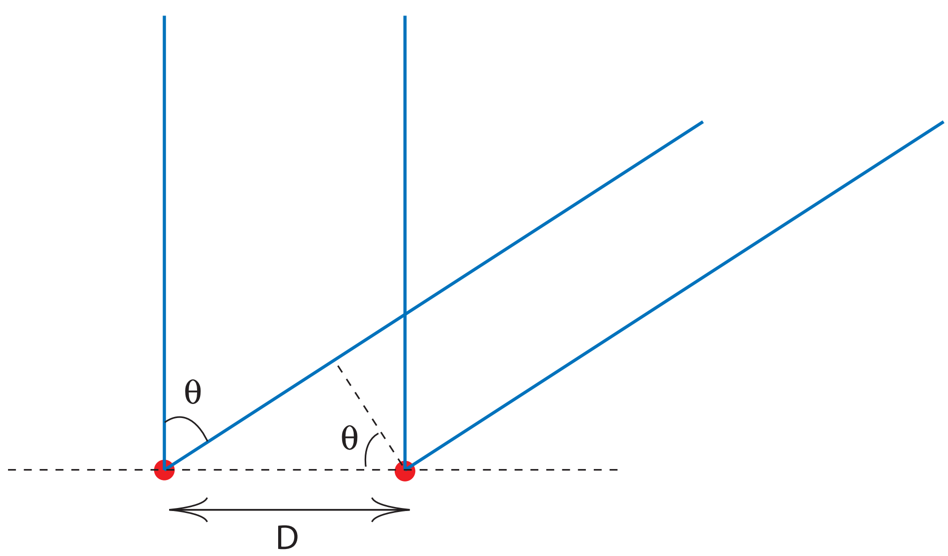
$$n\lambda = 2d\sin\theta, \tag{32}$$

where n is some positive integer, d is the spacing between atomic planes, and θ is the angle shown in the figure below. One can see that the Bragg relation is essentially just a fancy way of saying that constructive interference occurs when the path difference between the two rays bouncing off adjacent planes is an integer multiple of the wavelength.



The Davisson and Germer experiment is different in that the incoming beams of electron waves are at normal incidence (see diagram). Thus, we need a modified Bragg relation for this new geometry. If we imagine the electrons scattering off the surface layer of atoms, constructive interference occurs when (as usual) the path difference is an integer multiple of the wavelength:

$$D\sin\theta = n\lambda. \tag{33}$$



Putting everything together, we get a lattice spacing of

$$D = \frac{nh}{\sin\theta\sqrt{2qm\Delta V}}. \tag{34}$$

Plugging in $n = 1$ and $\theta = 50$ gives $D = 0.218\,\text{nm}$. This is in good agreement with the data from X-ray measurements.

The treatment we have given here is arguably somewhat simplistic, and the only justification for some of the assumptions we have made is that our analysis gives the right answer. For an examination of some of the more subtle aspects of the problem (*e.g.* electrons travel at a slightly different velocity when they're *inside* a crystal compared to when they're in a vacuum. Does this change things?), see Section 2-5 in "An Introduction to Quantum Physics" by French and Taylor.

Problem 6. (15 points) Single-slit Diffraction and Uncertainty

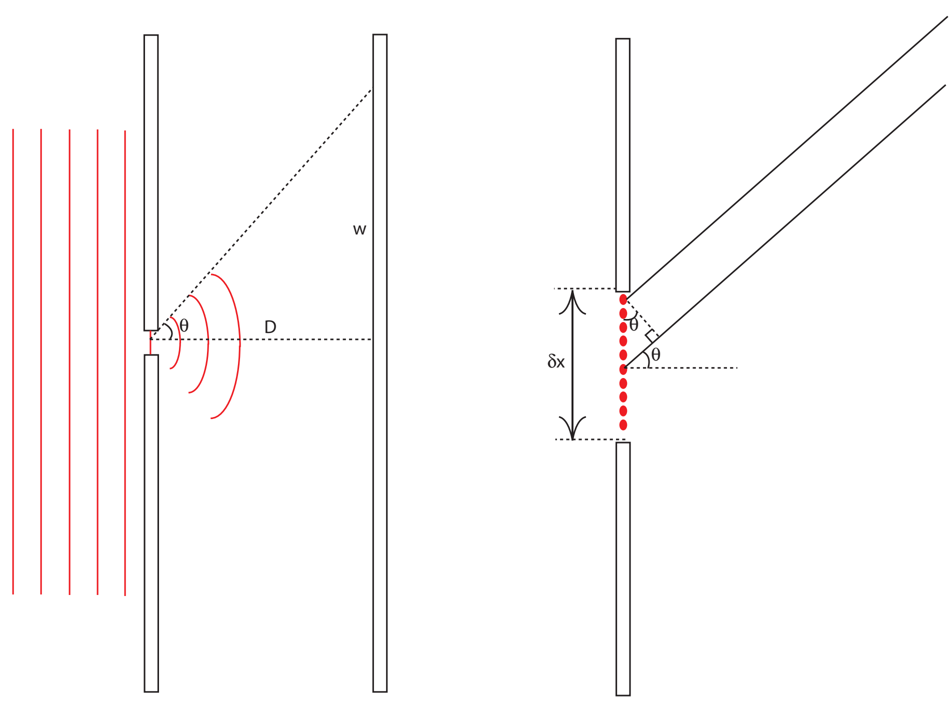
(a) **(5 points)** From the diagram, one sees that the pattern of the screen has a destructive minimum at the angle where each emitter can be paired up with exactly one other emitter such that the path difference between waves emanating from them is exactly $\lambda/2$. From the geometry of the diagram, this is equivalent to saying

$$\frac{\delta x}{2}\sin\theta_{edge} = \frac{\lambda}{2} \quad \Rightarrow \quad \theta_{edge} = \frac{\lambda}{\delta x}, \tag{35}$$

where in the last step we invoked the small angle approximation $\sin\theta \approx \theta$. Since $\tan\theta_{edge} = \frac{w}{D}$, we can eliminate θ_{edge} in favor of w and D, giving

$$w = \frac{\lambda}{\delta x}D, \tag{36}$$

where we have again used a small angle approximation (this time it's $\tan\theta \approx \theta$).



(b) **(5 points)** The photons with the greatest transverse momentum will be those at the edge of the pattern. For such photons, their travel direction is given by $\theta = \frac{\delta p_x}{p}$, which combines with $\theta \approx w/D$ to give

$$\delta p_x = \frac{w}{D}p. \tag{37}$$

(c) **(5 points)** Combining $E = hc/\lambda$ and $E = pc$ gives the standard de Broglie relation $p = h/\lambda$. From above, we have $p = (D/w)\delta p_x$ and $\lambda = (w/D)\delta x$, which gives the Uncertainty Principle when substituted into the de Broglie relation:

$$p\lambda = h \quad \Rightarrow \quad \delta x \cdot \delta p_x \sim h. \tag{38}$$

8.04 Quantum Physics I
Spring 2013