BOLESLAW WYSLOUCH: Let's get started. So today hopefully will be a busy day, with lots of interesting insights into how things work. We talked about coupled oscillators last time. We developed a formalism in which we can find the most general motion of oscillators.

So let's remind ourselves what are the coupled oscillators. Coupled oscillators, there are many examples of them, and they have more or less the following features. You have something that oscillates-- for example, a pendulum. You have to have more than one, because for coupled oscillators you have to have at least two.

So let's say you have two oscillators. So each of them is an oscillator, which in, for example, in the limit of small angles, small displacement angles, undergoes a pure harmonic motion with some frequencies. And then you couple them through various means. So for example, two masses connected by a spring is an example of a coupled oscillator.

We could have two masses on a track and another track, also connected by several springs. This is also an example of a coupled oscillator. Each of those masses undergoes harmonic motion, and they are connected together such that the motion of one affects motion of the other.

You can have slightly more complicated pendula. For example, you can hang one pendula from the other. Each of them-- again, in the limit of small oscillations-- will undergo harmonic motion. And they are coupled together because they are supported one on top of each other.

And you can have-- we have another example of two tuning forks sitting on some sort of boxes. Each of them was an oscillator, with audible oscillating frequency, and by putting them next to each other they coupled through the sound waves transmitted through the air. So one of them felt the oscillations in the other one. This was an example of coupled oscillation. Two masses and the thing.

You can build oscillators out of electronics. Some capacitor and inductor together, with a little bit of-- maybe without resistors. You have two of those. They constitute a coupled oscillator if you put a wire between them. So there are many, many examples.

And of course, these are all examples in which you have two oscillating bodies, but it's very easy to have three or more oscillating bodies. Then basically the features of the system are the same, except the math becomes more complicated, and we have more types of oscillations you can have.

And there's a couple of characteristics which are the same for all oscillating systems. And it's very important to remember that we are learning on one example, but it applies to very many. Number one, any motion-- I can maybe summarize it here. So if you look at the motion of an oscillator, you can have-- let's say arbitrary oscillation. Arbitrary excitation. Excitation means I-- I kick it in some sort of arbitrary mode. I just come in and set up some initial condition such that things are moving.

And motion in this arbitrary assertion is actually-- looks pretty chaotic. It looks pretty variable, changing. It's difficult to understand what's going on. So And it clearly doesn't look harmonic. Non-harmonic. There is no obvious single frequency that is driving the system.

If you look at amplitude of the objects here-- for example, two pendula, pendulum one and two. At any given moment of time they are oscillating, there's a characteristic amplitude. But what we saw is that motion changes, looks like things are flowing from one to the other. One of them has a high amplitude. After some time, it cools down, the other one grows. So the amplitudes are changing in time. So they are variable. Are variable.

And also, we didn't calculate things exactly, but you know from study of a single oscillator that if the things are moving, it has a certain amplitude, there's certain energy involved-- with some potential, some kinetic-- and it's proportional to the square of amplitude. So it's clear that energy is moving from one pendulum to the other. This one was oscillating like crazy. So all energy was sitting here. After some time, this one stopped. So its energy is zero. And the other one was oscillating like crazy.

So the energy's flowing from one to another. It's not sitting in one place, but it's flowing. This one has lots of energy right now, but now that one is picking up. So the energy-- you see the energy flowing here. And this one will eventually stop-- well, this is a pretty crappy oscillator, but it will eventually stop, and this one will have all the energy.

And this is, again, characteristic in every system. We can see energy flowing around from one to the other, growing, stopping. So it's-- in general, in the most general case, it's a complicated system. Energy is migrating between different masses. However, every single one of those coupled oscillating systems has a magic. There's a magic involved, namely the existence of normal modes. Every single coupled oscillator system has normal modes, and those modes are beautiful.

Those modes are-- everything is moving in sync. So this is normal mode excitation. There's a very special way, a special setting of initial conditions, that leads to the-- that results in a pure harmonic motion. So this is a harmonic motion, with a certain frequency omega, characteristic frequency for this particular motion. The amplitudes remain fixed.

Once you set initial conditions, you get it moving, everything is moving, simple harmonic motion means its amplitude is constant. So if I-- and remember, for example, this system. It was something like this. Symmetric or antisymmetric motion. And if not for the friction, the amplitudes would remain constant forever, if it will be a perfect oscillator.

So amplitudes-- in fact, it's not amplitudes themselves, but amplitude ratio. The ratio of amplitude between the different elements in the system is constant. So in a sense, every harmonic motion has a characteristic shape. And then by-- since everything is constant, nothing changes, this energy stays in the place it is. So energy is-- once you put energy to mass number one, mass number two, mass number three, the energy sits there. The energies are constant, as the system undergoes harmonic motion. Energy does not migrate.

So this is a very nice-- and there is another beautiful feature, that any arbitrary excitation can be made out of some linear sum-- sum of normal modes. Linear sum, of superposition of normal. Any arbitrary excitation with all its complicated motion can be made into some of normal modes.

So since normal modes are easy and simple and beautiful, the description of motion of any coupled oscillator, the best way to approach it is to decompose it, to find all possible normal modes, and then decompose the initial condition to correspond to this linear sum of normal modes. Once you know the normal modes, you add them up, and then you can predict exactly the motion.

And this is what we've done. So we have a-- we have introduced a mathematic mechanism in which we put all the information about forces and masses in the system in some sort of matrix form. In our example, it was a two by two matrix, but if we have three masses or four masses, the dimensionality of the matrix will have to grow. But the equation will remain the same.

So this equation of motion, we rework it a little bit. Since we are looking for normal modes, we know that normal modes occur with this one single frequency. So we postulate an oscillation with a frequency. We plug it in. We obtain a simple algebraic equation. Doesn't have any time dependence, doesn't have any exponents. It's a simple algebraic equation, basically a set of linear equations, which we can solve and find the eigenvalue, or the characteristic frequency for normal modes.

And you can show that the number of those frequencies in general is equal to the number of masses involved in the system. And you solve it, and then once you know the characteristic frequencies, then you can find shape, you can find the eigenvectors. What is the ratio of amplitudes which corresponds to the mode.

And in case of our two pendula, there are two of such things. One is where both amplitudes are equal, and this corresponds to oscillation in which two pendola are moving parallel to each other, with a spring being-- not paying any roll. So this is one mode. And then amplitude is-- as I said, any given moment is the same, so the ratio is 1.

And then you have a motion in which the two pendula are going against each other. So any given moment of time, they're in their negative position, so the ratio is minus 1. The motion of one of them can be obtained by looking at where the first one is and multiplying by minus 1. So these are the two modes, and any arbitrary-- any complicated, nasty excitation with things moving around is a linear sum of the oscillation.

So we know that. We've worked it out. We used this example. And by the way, today, we'll be using two examples-- one which is the same thing with two pendula and the spring, and the other one with two masses, or maybe later three masses. And the exact values of coefficients in matrix k are different in two different cases. But in all types of other motion, the shape of motion, the behavior of the system is identical. So the solutions to the two cases are identical. The difference is basically numerical in how the spring constants and masses come in. So we can in fact treat those two systems completely the same. So I'll be jumping from one to another, but we don't have to worry.

But let's now look on the system. So what we are trying to do today is, we are trying to apply external force so we'll have a driven coupled oscillator. And I assume that you know everything about driven oscillators. So the idea was that you come with an external . In 8.03, we assumed that this external force is harmonic force. So there's a characteristic frequency which is given by external-- let's say by me. It has nothing to do with normal frequencies of the system. It's an external frequency, omega d, which I apply. Driven frequency.

And then I look at how the system responds. And I look for steady state oscillations-- the ones where everything oscillates with the same driven frequency-- trying to look for solutions. And as you know from a single oscillator, what we were calculating is what is the the response of the system? What is the amplitude?

And the certain frequencies that-- you wiggle it and the system doesn't do anything, but if you apply a certain resonant frequency, then the response is very large. The system starts moving like crazy, et cetera. And the same type of thing will happen here, except that we have multiple frequencies. So there will be a possibility of a resonance for several frequencies. All right?

So let me quickly set this up. Just-- yeah. Doesn't matter. So there were some-- let's just start working on the example. So just a reminder, this is our system. A pendula of some length L. There are two identical masses, M. There is a spring of constant k. They are all-- and for simplicity, we assume that we are all in Earth's gravitational field. So we don't have to worry about traveling to Jupiter or the moon.

And-- except that the difference will be that we apply an external force to one of those masses. How, it doesn't matter, but there is an external force F-- F with subscript d, which is equal to some-- it has some amplitude F0 cosine omega d times t, along the x direction. And this is applied to mass one.

OK. And there is a little bit of just a warning. We will be assuming that there is no damping in the system. For the single oscillator, there was always a little bit of damping. So between you and me, remember there's always a little damping. So in case we need damping-- it will come in and will help us, but if we try to use damping in calculations, calculations become horrendous. So for the purpose of calculations, we will ignore damping. It'll get some. But if things go bad with the results, like dividing by 0, then we will bring in damping and say no no, it's not so bad. Damping helps you. We are not dividing by 0. OK?

So let's write those equations of motions. Equations of motion. So we have-- so the forces and accelerations on mass one is the same as before. There was a spring. There is mg over l.

That's the pendulum by itself. Depending on position x1. There is the influence of a spring, which depends on where spring number two is. And, plus, there is this new driven term, F0 cosine omega d times t, where omega d is fixed, arbitrary, externally given. So both F0 and omega d are decided by somebody outside of the system.

Now, the second mass M X2 dot dot, is-- actually has feels position of x1, through the spring. And there is this-- its own pendulum effect plus a string, depending on position x2. Interestingly, there is no force here, because the force is applied to mass one. So mass two a priori doesn't know anything about the force. But of course it will know through the coupling. Yes? Questions? Anybody have questions so far?

So it's the same as before, with the addition of this external force. Again, this is writing all coordinates one by one. We immediately switch to matrix form. We write it MX double dot, where X is the same as we defined before, minus KX. I think I will stop writing these kind of thick lines. But for now, let me-- F cosine omega d times t.

So this is now a matrix equation for the vector XD. And let's remind ourselves what those matrices are. Matrix M is M 0 0 M. This is just mass of the individual systems. We use M minus 1, which is 1 over M, 1 over M, and diagonal 0 and 0. So this carries information about masses, inertia of the system. Matrix K contains information about all the springs in the system, and some pendula effects. So we have a k plus mg over l, minus k, minus k, k plus mg over l.

And now there is this new thing, which is this vector F. Vector F is equal to F0 0 cosine omega d times t. So this is in a vector form, this external force, which is applied only to mass number one. OK?

So these are the elements which are plugged in. So now the question is, what do you want to do with this? So we have the equation of motion. And so what do we do with this? So there are two steps that we have to do. Number one, we have to remind ourselves what are the normal modes of the system, in case-- because we will need-- the information about normal modes will come in as-- into solutions for a driven motion.

So let's remind ourselves what this was. Well, this was a solution. I'll just rewrite it very quickly such that we have it for the record. It should fit here. Now let's try. So there were two solutions. There was omega 1 squared, which was equal to g over l. And the corresponding normal mode was a symmetric one. It was 1, 1. OK. So this was one type of solution, where the two masses were moving together.

There was a second frequency which was equal to g over l. The square of it was equal plus 2k over m. And this was the characteristic normal frequency for the second type of oscillation, which you can write it 1, minus 1. And the criterion for when we were looking for solutions, we would find them by calculating the determinant of this two by two matrix. It was the determinant of m minus 1 k minus omega squared times unit matrix was equal to 0. So this was the equation that had to be satisfied for frequencies corresponding to normal modes with zero external force.

Interestingly, if you do the calculations, it turns out you can-- algebraically, you can write-- after you know the solution itself, you can write it in a very compact way. So this determinant can be written in the following way-- omega squared minus omega 1 squared, times omega squared minus omega 2 squared. And this is-- the condition was zero. And you see explicitly that this is a fourth order in frequency equation, fourth order frequency, which is 0 for omega 1 and for omega 2. In a very explicit way. So this is a nice, compact form of writing this particular eigenvalue equation.

And again, as a reminder, the motion of the system-- the most general motion of the system with no external force was a superposition of those two oscillations, which we can write as some sort of amplitude-- 1, 1 cosine omega 1 t plus phi 1, plus beta 1, minus 1 cosine omega 2 t plus phi 2.

So this is oscillations of two different frequencies. This is the shape of oscillations, the relative amplitude of one versus the other. And then there's the overall amplitude alpha and beta, which has to be determined. And then there are arbitrary phases. So there are in fact four numbers, which can be determined from four initial conditions. So typically two positions for the two masses, and two initial velocities for two masses. So everything matches. So this a so-called homogeneous equation. Homogeneous solution.

What about driven solution? Driven solution, as we remember from a single oscillator, results in a motion in which all the elements in the system are oscillating at the same frequency, and that's the driven frequency. It's a fact. I come in, I apply 100 Hertz frequency, and everybody oscillates on the 100 frequency. That's the solution for a driven oscillating system.

And we saw it for a one-dimensional oscillator, and we will see it here as well. There's one frequency, omega d. So we will be now looking for a solution which corresponds to the oscillation of the system with this external frequency, which a priori is not the same as one of the normal frequencies.

So the complete motion of the system consists of two parts. One is this homogeneous self-oscillating motion with two characteristic frequencies. And there will be a second type of motion, which is a driven one. So how do we go about solving that? So equations of motions of course will be the same. The solution, the way that we solve it will be very similar. So lets try-- start working. Maybe we can work on those blackboards here.

So what is going on? So we know that if we apply external frequency omega d, everybody in the system, all the elements will be oscillating with the same frequency. So we can then introduce a variable Z, which will be defined B e to the i omega d t. This will be the oscillating term. And this will be the amplitude of oscillation, which we'll try to make real for simplicity.

And then we plug this into the equation of motion, which is listed up there on the screen. So the equation of motion is Z dot dot plus M minus 1 K times Z is equal to now M minus 1 force e to i omega d t. You see our external force is F cosine omega d t, with a vector 1, 0. But of course, in the complex notation, this is exponent.

So this is the challenge, what we would like to have. And we assume that all the elements in the system-- position, acceleration-- oscillate at the same frequency omega d. If you do that, then the equations become somewhat simpler, because the oscillating term drops out. So when you plug this type of solution into here, what you get is minus omega d squared-- that's from second differentiation with respect to time-- plus M minus 1 K, multiplying vector B e to i omega d t. This must be equal to M minus one F e to i omega d t. This is vector B, this is vector F. And there is this oscillating term. But both sides oscillate at the same frequency. That's what we assume. So we can simply divide by this, and we are left with an equation that equates what's going on in the oscillating system with the external force.

So now, let's see here what is known and what is unknown in this equation. M minus 1 K carries information about the construction built of the system of accelerators. Strength of springs, masses, gravitational field, et cetera. So this is fixed. This is given. Omega d is the external driving frequency, and it's also given. It's a number. I said this is externally given. I just set it at some computer. Say 100 Hertz, and it's driven at 100 Hertz. So we know that. We know exactly what this number is.

External force, we know what it is. We defined it. It's F0. We know what its magnitude-- so everything is known except for vector B. And vector B are the amplitudes of oscillation-- remember, everything oscillates at omega d-- of mass one and mass two. So in general, if I apply external force, this guy will oscillate with some amplitude. That guy with some amplitude, a priori different. And this will be B1, this will be B2. And we don't know that at this stage.

So this equation will allow us to find it. And it is possible because-- this is actually a very straightforward equation. It contains-- actually, to be very precise, I have to-- this is a number, this is a matrix. So I have to put a unit matrix right here. So it's omega d times unit matrix plus this matrix that carries information about the system.

And so we can write this down again in some sort of more open way, for our specific case. So this will be k over m plus g over l, minus omega d squared, minus k over m, minus k over m, k over m, plus g over l, minus omega d squared.

So this is this matrix here. This matrix is applied to vector B, which is our unknown. Let's call it B1 and B2. These are the amplitudes of oscillations of individual elements in our system. And this is equal to m-- the inverted mass matrix times vector F, which-- without its oscillating part, which is simply F0 over m and 0.

All right. So this is the task in question, and we have to find out those two values depending on these parameters and the strength of force, et cetera. So this is actually not a big deal. It's a two by two equation, two equations with two unknowns. We solve it, and we are done.

However, we want to learn a little bit about slightly more general ways of calculating things. So let's call this one matrix E, with some funny double vector sign. Let's call this one vector B, and let's call this one vector D, because we will use this-- use it later. And what we are trying to do is, we are trying to use the so-called Cramer's rule to find those coefficients B1 and B2. And for some historical reasons, 8.03 really likes Cramer's rule. I like MATLAB or Mathematica. I just plug things in, and it crunches out and calculates. But it turns out that for two by two, you can always do it quickly. Even for three by three, if you just sit down and do it, you can actually work it out. It's not scary. By five by five-- but even four by four, I'm sure you are mighty students who can just do it in the exam. I have never seen an 8.03 exam with four masses, unless they're general questions. But three-- well...

All right. So do we go about finding this B1 and B2? Because, again, this is a simple two by two question. So maybe just to again bring it even closer to what we are used to, let me just quickly write this down as a set of two by two equations. So there is a coefficient here, k over m plus g over l minus omega d squared, which is-- this is a number, times B1 minus k over m times B2 is equal to F0 over m minus k over m B1 plus k over m plus g over l minus omega d squared is equal to 0-- times B2 is equal to 0.

So you see two equations with two unknowns. Couple of coefficients, all fixed. You can eliminate variables. You can calculate B2 from here, plug it into-- you can work it out if you want to. However, there is, again, a better way. It's Cramer's rule or method. Should have known if it's method or rule. Rule. Right.

And so the way you do it is the following. So you look at those questions-- you calculate all kinds of determinants, and by taking the set of two equations and plugging into-- replacing columns in the matrix. So B1, what you do is you take the original matrix, which is here, and you replace the first column of the matrix with vector B. So you-- no wait, with-- sorry, with vector D. Take this matrix, and you plug in this.

So what you do is-- so it turns out-- so B1 can be explicitly calculated, but taking the determinant of the first column replaced, F0 over M0, and keeping the second column, which is minus k over m. m and then k over m plus g over l minus omega d squared. So this is-- you calculate the determinant of this thing, where-- original matrix with the first column replaced. And you divide it by the determinant of the original matrix. Let's call it E. So you calculate this determinant again for the frequency omega d.

So this can be written very nicely, in a very compact way. This determinant is easy. It's just this times that. So have 0 over m multiplying k over n plus g over l minus on I got the squared remember this is a given number divided by n Here comes this nice compact form for the determinant, which is omega d squared minus omega 1 squared, times omega d squared minus omega 2 squared, where omega 1 and omega 2 were the normal mode frequencies. Yes?

AUDIENCE: Where are you getting the minus k in the [INAUDIBLE]?

BOLESLAW WYSLOUCH: This one?

AUDIENCE: Yeah. [INAUDIBLE]

BOLESLAW WYSLOUCH: This one? This is the second column. See? I'm taking-- so this is the first column, second column. I take the first column, I replace it with driven equation-- with a solution. I plug it here. So I have F0 for M0. And I keep the second column. All right? That's for the first coefficient. For the second coefficient what you do is, you put a driving term here and you keep the first column. All right?

So this is actually an explicit solution for B1. This is magnitude of oscillations of the first element. And you can do the same thing for B2. And I'm not trying to prove anything, I'm not trying to derive anything. I'm just using it. And I'll show you a nice slide with this to summarize. So B2 is the determinant of-- I keep the first column. It's k over m plus g over l, minus omega d squared, minus k over m. That's the first column. And I'm plugging in F0 over M here, and 0 here.

So this is-- and divided by omega d squared minus omega 1 squared times omega d squared minus omega 2 squared. That's the determinant of the original matrix. And this one is also very simple It's this time this is 0. I have minus that. So I simply have F0 k over m squared divided by omega d squared minus omega 1 squared, omega d squared minus omega 2 squared.

All right. So we have those things, and also what? Do you see anything happening here? Yeah, there are some numbers, but what do they mean? What does it mean? Yes, we can calculate it. You can trust me. These are the-- I'm not sure that you can trust it, but most likely these are good results. And so we know the oscillation of the first mass, oscillation of the second mass as they are driven by the external force.

Now, one of the interesting things to do is to try to see what's going on. One of the-- when we talked about normal modes, the ratio of amplitudes carried information. Remember, we had those two different modes. Either amplitudes were the same, or they were opposite sign. So let's ask ourselves, what is the ratio of B1 and B2? So let's just divide one by the other. So let's do B1 over B2. Let's see if we learn anything from this. If you divide B1 over B2, this bottom cancels out, and I have k over m plus g over l minus omega d squared over k over m.

And-- yeah. So now comes the interesting question. This omega d can be anything. So let's say omega d is-- so we can analyze it different ways. So for example, when omega d is-- you can look at small, large, and so I can compare it. But one of the interesting places to look is, what happens when omega is very close to one of the-- to the characteristic frequencies?

Because, remember, when we analyzed a single driven oscillator, the real cool stuff was happening when you are near the resonant frequency. Things, you know, the bridges broke down, et cetera. So let's see if we can do something similar here.

Now we have two choices. We have omega 1, omega 2. So let's see what happens if I plug in omega 1. Omega d being very, very close to omega 1. Let's say equal to omega 1. Omega 1 is-- omega 1 squared was g over l. So if I plug omega 1 here, I have k over m plus g over l. So I have k over m plus g over l, minus g over l, divide by k over m, which is equal to what? Those two terms cancels. k over m, it's plus 1.

That's interesting. So if I drive at a frequency which corresponds to omega 1-- and omega 1 was the oscillation where both masses were going together. So the characteristic normal mode had the ratio of two masses equal to one. And here I'm getting the system to drive at this type of mode. Again, I have-- the driven amplitudes are the ratio is equal to one.

So what happens if I drive at omega d close to omega 2? Omega 2 squared was equal to g over l plus 2k over m. If I plug it in here, I get that the ratio is minus 1. Again, the ratio is strikingly similar to the ratio of the normal mode corresponding to frequency omega 2. So it's like I'm inducing those oscillations.

So what does this all mean? There's, by the way, a little catch here for all of your mathematicians. What happens to equations if I set omega d equal to minus 1-- to omega 1, for example? I just plugged it here, and nobody screamed. But there was something fishy about what I did. Yes?

AUDIENCE: --coefficient [INAUDIBLE]

BOLESLAW WYSLOUCH: If you took--

AUDIENCE: Oh, sorry. [INAUDIBLE]

BOLESLAW WYSLOUCH: Exactly. So the ratio of the two was one, but both of them were infinite. So infinite divided by infinite equals what? I mean, this happens. So what's going on? Why can I do it? One-- we should not really scream. Damping. Exactly. This is where the damping comes in. So the amplitude is enormous, but it's not infinite, because there's always a little damping. The system will not go to infinity.

So in real life, there's a little term here that makes sure things don't blow up completely. There's a little damping here. Yes?

AUDIENCE: Does it at all matter-- also the fact that those equations are inexact in the first place, because we had made theta smaller--

BOLESLAW WYSLOUCH: No. That's not-- no. This doesn't actually matter. It's the absence of damping that makes things look nonphysical.

AUDIENCE: But as the frequency-- as the amplitude increases, when we're in resonance, eventually those equations wouldn't hold any longer, and perhaps--

BOLESLAW WYSLOUCH: Yeah, that's right. But you could-- that's true. That's true. But you can come up with, for example, an electronic system which has a huge range of-- enormous range of possibilities. And then-- or of amplitudes. Many, many-- so the damping is much more important in that. So in reality, there is some damping here and so forth.

All right. So why don't we do, now, the following. So let's try to see how this all works out. First of all, such that we can get started, I will make a sketch for you. I'll calculate these formulas-- just a second-- and display you as a function of frequency, such that we can analyze what's going on.

So where is it-- OK. It's still slow. All right. So this is what those-- OK, so let's say-- I don't know which is which, but let's say B1 is the red one, B2 is the blue one, or vice versa. It doesn't matter. These are the numbers which I plug in for some values for some system. So we see that-- and this is as a function of frequency.

So first of all, you see a characteristic frequency around one, characteristic frequency around three on my plot. And in the region in the vicinity of frequency number one, you see that both the blue and red, the individual amplitudes are basically close together. So the ratio is close to one. If you look at this plot, you should believe me that it's plausible that if you are very close to the frequency, basically the red and blue will move together.

If you go around the second frequency, you see that red goes up, blue goes down, or vice versa on the other side. So the ratio is minus 1. So this plot actually carries in formation. And in fact, what you see also is that there is some sort of resonant behavior. So the amplitudes are enormous if you are close to any of those characteristic frequencies, but they're much smaller if you're further out. There is some motion, but not as pronounced as when you're at the right driving frequencies.

All right. So let's try to see it. Why not? So let me go to another system-- a system which consists of two masses, has the same type of behavior, slightly different parameters. There is no g here, but everything looks the same. It's just much easier to show. And I can remove most of damping. And you'll see there are again two modes, one which is like this-- that's number one, that slow motion. They move together. And the other one, which is like this, where the amplitudes are minus 1. This is the frequency number two.

So now let's try to drive it. How do I drive it? I have some sort of engine here which is applying frequency. So let's start with some sort of slow motion. So you see they are moving a little bit. Very small, minimally. Just a tiny motion. But they're kind of together, more or less, right? Slowly, but together. And this is what-- this is this area here. I don't know if you see that. This is this area. I'm driving at a very slow frequency. I'm somewhere here. The two masses kind of go together, but very slowly.

So let me now crank up the frequency and try to be in the region of oscillation. So you see? All I did is I changed frequency. The effect is enormous. I'm somewhere here now. You see? Enormous resonance. And very soon, I will hit the limit. The system will break. OK, so we are somewhere here. I'm driving it. Interestingly, this really looks like a harmonic motion of first type. There is no other things.

OK, so now let's swing by and get to this area. So all I'm doing is, I quickly change frequency to- this one. So now what you see is that there were some random initial conditions, so we have a homogeneous equation going, but the driven is coming in. All I did is I changed frequency. And suddenly the system knows that it has to go like that. Isn't that cool?

So this is the region here. And all I'm doing is I'm bringing the amplitude up, because this is close to zero. And then I'm keeping the ratios close to the characteristic modes. So I think-- to be honest, this is one of the coolest-- all I'm doing, just changing frequency. And the system just responds and starts going with a resonance of one particular mode.

So imagine a system that has 1,000 masses, and you come in with 1,000 frequencies. You tune one frequency, and suddenly everything starts oscillating in one go. And imagine you have multiple buildings, each with different frequency, and there's an earthquake. And the frequency is of a certain type, and one building collapses, and all the other ones are happily standing. Why? Because the earthquake just happened to hit the frequency that corresponded to one of the normal frequencies of that particular building.

And it's an extremely powerful trick. It fishes out normal modes through this driving thing. And we are able to calculate it explicitly. So now what I will do is, I will modify the system and I will make it into a three mass thing, which will have a somewhat more complicated set of normal modes. And then I will show you that I can in fact go with three different frequencies, and pull out those even complicated modes.

So this will be it. So this is a three mass system. Now before, since we didn't calculate it, what I will do is, I'll go to the web and I will pull out a nice example. Let me go to my bookmarks. Normal modes. So this is a nice applet from Colorado. And you can-- I suppose preso ENG will send you links, et cetera. You can simulate-- you can do everything with it. So it has two masses. It has different amplitudes, different normal modes.

And you can see nothing happens. So I have to give it some initial condition. Sorry, I have to change polarization. Where is polarization? Here. I give it some initial condition. So this is basically what you just saw. I'm just demonstrating to you that this applet looks the same as our track. So this is you can see normal modes. It's a combination of normal modes. There's one which is first frequency, second frequency. This is first normal mode. This is second normal mode. You can very quickly see what happens. So this is what we just looked at. This is what we calculated, more or less, and so on.

Now I want to show you three masses where things are somewhat more complicated. In general, three normal modes. For the three mass elements, the first normal mode is like that. All the three masses move together. And slightly different-- the ratio of amplitudes is slightly different. The second mode of operation is actually quite interesting. The central mass is stationary, and those two are going forth and back, like this.

And then I have a third frequency where the middle one is going double the distance, and the two other ones are going up. So this is the third normal mode.

All right. So this is the system which we now have standing here. Let's quickly see if it works in reality. So this is the first-- so this is the first mode. This is the second one. All right. And the third one will be-- Sometimes I do five of them, and then it's really difficult.

OK. But-- so we have a computer model, we have a real model. Let's now do the calculation of the frequencies, the ratios, such that we can see what happens. So I'm coming here, I'm changing mass to three. I'm running my-- the terminal calculating thingy. OK. It's very slow. It's busy, busy, busy. Imagine-- OK. Spectacularly slow. Where is it? I hope it's not-- oh, here it is.

OK, so this is what's coming out. So this is the same calculation as we did, except for three masses. So what do we have here? Where's my pointer? So we have, again, three characteristic frequencies, we have three masses, and the same type of behavior. See, if you are far away from resonance, if you have very low frequency, everybody goes together.

I haven't shown you this one here, which is also interesting. I'll show you in a second. And then-- so presumably if you are close to the first frequency, you see all three of them go together. And this is the first mode. So I should see, if I set the proper frequency, the thing should respond in mode number one.

This is the one where two of them go opposite to each other, and the red one is stationary. It doesn't move. And then you have those things where they're kind of more complicated. It's difficult to read them from here. And I can do it for more masses, et cetera. So generally it's calculable. It can be calculated and can be actually demonstrated. So let's try it.

So-- 32. So there's this magic frequency number one. I'm setting frequency by turning a knob. That's omega d. I'm a supervisor of this operation. It stops because of other reasons, but it will continue. Then I go to 56.

By the way, remember that every-- this is the particular solution. This is a steady state distillation with omega d. But we also have all those homogeneous solutions, which have to die down with damping. Remember, it's a combination of homogeneous plus particular. So the motion is actually a little bit distorted because we have this homogeneous stuff hanging around. But hopefully, if I can start it with little homogeneous stuff, it will be better.

So you see? Pretty cool. Almost there. It's almost in assembly. Then it kind of stops. You see? I get two of those going forth and back, more or less, and this one going. I could probably tune the frequency a little bit higher or lower. I'm not exactly at the right place, but I'm close.

And now let's go to the last one, which is 68 according to my helpers here. You see? This one goes opposite phase, and those two more or less together. Then they keep going. See now, those two move a little bit forth and back, but they are in phase. They move together. The ratio is 1. And this one-- the ratio is minus 2.

Right? Make sense? That's the beauty. You drive it at some frequency, and those normal modes pop out. It's actually very, very cool. And as I said, you encounter those type of behaviors very often. Sometimes you drive a car and something starts vibrating, it's just because the car driving on the road creates a frequency, provides a driving frequency which corresponds to oscillation frequency or some piece of-- old car. Usually it happens in old cars.

So I think that's the message we can-- and we have all the machinery to be able to do it. We can set up any matrix at K, which has information about all the forces acting on anything, and we can set matrix M with the masses. We can put it all together, we can find normal modes, and then we can use Cramer's equation to take care of the arbitrary external forces.

And what comes out, just as a-- for summary, for future reference, the oscillation of the system is-- this is conveniently written. This is vector X. In general, this homogeneous solution this plus the particular solution, which is plus vector B, which is very important. Vector B depends on the driving frequency. Those amplitudes of a particular solution during emotions are dependent on driving frequency. Cosine omega d times t.

So in the most general situation, we have some homogeneous solution here, and there is this driven solution which we observed in action, with proper amplitudes. So in fact, what you've seen is the sum of both, because this depends on the initial conditions.

Now, in reality, as with a single oscillator, this homogeneous equation, there's always a little damping, which we ignore it. And the damping comes in, and it only affects the homogeneous solution. So this part will eventually die down, whereas a driven solution is always there. There's external force that is driving the system forever and ever. So this part, this steady state or particular solution will remain forever, because there's an external source of energy which will always provide it. So these guys will die down.

And of course, because of damping the exact value of coefficients B will be slightly modified, because as you know from the from a one oscillator example, the presence of damping actually slightly modifies the frequency. Whereas here, we-- for simplicity-- if we introduce damping here, those calculations are really amazing. So we don't want to do it.

All right. Any questions about it? Yes.

AUDIENCE: If we were doing Cramer's rule with a three by three matrix, would we only replace the column that corresponds to the B that we're trying to find, and then keep the other two?

BOLESLAW WYSLOUCH: Yes. So it's always-- you'll be doing always that. In fact, I should have some slides from Yen-Jie on Cramer's rule. Let's see. OK. So this is some reminder of last time. So this is Cramer's-- there's Mr. Cramer. So this is an example of what-- this is the two by two, three by three. OK? That's what you do. Question?

AUDIENCE: So it makes sense that the Cramer's rule [INAUDIBLE], but what does that mean for physical system?

BOLESLAW WYSLOUCH: Well, basically-- so the Cramer's rule is Cramer's rule. The question is what do you plug in? And what you plug in depends on the omega d. So it is true that if you insist on plugging in omega d exactly equal to one of the normal frequencies, then things blow up mathematically.

In reality, there is-- this is the situation of resonance. So as I discussed this before, in reality there is a little bit of damping. So those equations have to be modified. There will be some small additional terms here that will prevent this from being exactly equal to 0. So this will be a very large number. The amplitude will be enormous. If I would have a little bit more time, I'll fiddle with frequency, I could actually break the system, because those masses would be just swinging forth and back like crazy.

So you basically go out of limit of the system. So physically, there's always a little bit of damping. You do not divide by zero. On the other hand, it's so close that, for simplicity and for most of the-- to get a feeling of what's going on, it's OK to ignore it. Just have to make sure you don't divide by 0. So you can do this Cramer's rule with arbitrary omega d. Make sure you don't divide by 0, you solve it, and then you can interpret what's going on.

Again, Cramer's rule has nothing to do with physics. It's just a way to solve those matrix equations. As I say, you can do it anyway you want. Two by two, you can do it by elimination of variables. Five by five I do by running a MATLAB program. Anything you want. But for some historical reasons, 8.03 always does Cramer's rule. All right? And, yeah, it's useful, especially for three by three. All right?

OK. So I have to start a new chapter. I'm much slower than the engine, by the way. I don't know if you noticed. And that is the-- there's a very interesting trick that you can do which is of an absolutely fundamental nature in physics, which has to do with symmetry.

You see, many things are symmetric. There's a circular symmetry. There's a left and right symmetry. Example, two little smiley faces are mirror images of each other. There is some-- this thing is symmetric along this vertical axis. This one is symmetric around rotations by 30 degrees. That house seems to be symmetric along this way. This is part of our experiment in Switzerland, also kind of symmetric in the picture. The rotational symmetry, there's reflection symmetry, et cetera.

It turns out, if you have a system that is symmetric, then the normal modes are also symmetric. And there's a way to dig out normal modes just by looking at symmetry of the system. So let me explain exactly what this means. So let's take our system here-- OK, so we have one mass, the other mass. There is a spring here. This one is x1, this one is x2. If I take a reflection of a system-- let's say this mass is displaced by some distance. Some x2. This one's some x1. If I do the following transform-- I replace x1 with minus x2, and x2 with minus x1, this is mirror symmetry.

I basically flip this thing around. In other words, what I do here is I look at the system here-- hello-- and I go to the other system. Hello. Right? I did a mirror transform. I looked at it from this side, that side. Now, when I look at it I see the one on the left, one on the right. I call this one x1, this one x2. It's oscillating. You are looking at it, this is your x1, this is your x2. When I move this one, is it-- it's your negative x1. For me this is positive x2. This one is positive x2 for you. It's negative x1 for me. Do we see a different system? Does it have different oscillations? Does it have a different frequency? No. It's identical. They're completely identical. So the physics of those two pendula doesn't depend on if he's working on it or if I'm working on. That's the whole thing.

And this is how you write it mathematically. And if you have a solution which-- x1 of t, which consists of some sort of x1 of t, x2 of t. Let's say we find it. Now it's over there. We know alphas, betas and everything. Because of the symmetry, I know that for sure the equation which looks like this-- x1-- no, it's not x1. It's x of t. That's the vector x of t. I have another one with a tilde, which is identical functions, everything is dependent, except that this one is minus x2 of t minus x1 of t. And I know for sure that if this is the correct solution, this is also a correct solution. Why? Because he did x, and I did x tilde. But the system is the same. Completely identical.

And you don't have to know anything about masses, lengths, springs, anything like that. Just the symmetry. All right. How do you write it in matrix form? You introduce a symmetry matrix, S, which is 0, minus 1, minus 1, 0. And then x tilde of t is simply equal S, x of t. And we can check that. That's simple you just multiply the vector by 0, minus 1, minus 1, 0, and you get the same thing. Turns out-- and if this is symmetry, if this is a solution, this is also a solution. So we can make solutions by multiplying by matrix S.

So what does it mean? So let's look at our motion equation. The original motion equation was-- equation of motion was minus 1 k matrix times x of t. This is what we use to find solutions. Usual thing, normal modes, et cetera. Let's multiply both sides by matrix S. I can take any matrix and multiply by both sides. So I get here S X double dot of t. And of course, S is a fixed matrix, so it survives differentiation. And this is equal to minus S M minus 1 k x of t. Just multiply both sides by S.

However, if MS is equal to SM, and KS is equal to SK-- in general matrices, the multiplication of matrices matters. But it turns out that if the system is symmetric, if you multiply mass M by S, you just replace-- it will just change position of two masses. So nothing changes. Also, if the forces are the same, then multiplying mass S, you flip things. Nothing changes. And mathematically, it means that the order of multiplication does not matter. It means that they are commuting. And of course, M minus 1 S is equal to S M minus 1.

If this is the case, then I can plug it into equations and see what happens. So I can take this equation, and I can take this S here and I can just move it around. I can flip it with M1 position, because the order doesn't matter. So I can bring it here. And I can flip it with K, because the order doesn't matter. I can bring it here.

So after using those features, I get that S X dot dot is equal to minus M minus 1 K S X, which means that X dot dot-- remember, this was-- I'm using this expression. I'm just-- S times a variable x gives me X tilde. X tilde dot dot is equal to minus M minus 1 k X tilde. X tilde. Which basically proves-- this is a proof-- that x tilde is a solution. So if a system is symmetric, it means that it commutes-- that mass and K matrices commute, and you can-- and this means that this holds true. If I have one solution, the symmetric solution is also there. All right?

Let's say x-- yes?

AUDIENCE: So in the center equation, you introduced negative S. I didn't really get that.

BOLESLAW WYSLOUCH: So this negative is simply the-- Hooke's law. This is this minus sign here.

AUDIENCE: Yeah, but where did the S come from in the--

BOLESLAW WYSLOUCH: Oh, I multiplied both sides by S.

AUDIENCE: Oh, OK.

BOLESLAW WYSLOUCH: I just brought the S and I put it here. S, X dot dot, and S after-- minus commutes with S, so I kind of shifted my minus. But then I waited before I hit the matrices, because I wanted to discuss. OK?

So now comes the interesting question. Let's say X is a normal mode. Right? We have normal modes. Let's say X is a normal mode. It oscillates with a certain frequency. So I have X of t. Let's say it's equal to-- let's say it's a normal mode number one. Cosine omega 1 t. And we know that X tilde is also a solution. So what happens to mode number one when I apply matrix S?

So X tilde-- so matrix is a constant number. It's just a couple of numbers I just reshuffle things, et cetera. Try So if I have X, which is oscillating with frequency omega 1, if I multiply by some numbers and reshuffle things around, it will also be oscillating in number one. So it will be also the same normal mode. So if I take matrix S, I apply it to the normal mode, I will get the same normal mode, with maybe a different coefficient. Linear coefficient. Plus, minus, maybe some factor, something like that.

So if this is the solution, it means automatically that X tilde is proportional to A1 cosine omega 1 t. And the same is true for omega 2. So the only way this is possible, since cosine is the same in both cases-- matrix S to normal solution gives me normal solution with some sign. So the only way this can work, matrix S actually works on vectors, on A1. This is just an oscillating factor. So we know for sure that S A1 must be proportional. to A1. Similarly, S times A2 is proportional to A2.

So let's try to see with our own eyes if this works. So let's say S is 0, minus 1, minus 1, 0, times 1, 1 is equal to what? 0 minus 1, I get minus 1 here. This one, I get minus 1 here, which is equal to minus 1 times 1, 1, which is vector A. So vector 1, 1, which is our first mode of oscillation, is when you apply the matrix S, you get a minus 1 the same thing.

And similarly, if you do the same thing with matrix S-- so you see, the simple symmetric matrix consisting of 0s and minus 1s has something to do with our solutions, which is kind of amazing. So if I have 0, minus 1, minus 1, 0, I multiply by 1, minus 1, I get 1 here, I get minus 1 here. Just a moment. Something is not right. Something's not right. No, it should be--

AUDIENCE: [INAUDIBLE]

BOLESLAW WYSLOUCH: Hmm?

AUDIENCE: [INAUDIBLE]

AUDIENCE: It's 1, minus 1.

BOLESLAW WYSLOUCH: 1, minus 1. Yes. I don't know how to multiply here. I should be fine. OK. That's right. This is-- sorry, this is 1, because it's minus 1 times minus 1, and this is-- yeah, that's right. Which is 1 times 1, minus 1. So this is something that-- I get the same vector multiplied by plus 1. So this is, of course-- these are eigenvectors and eigenvalues. So the matrix S has two eigenvectors, one with eigenvalue of plus one, the other one plus 2. So we have an equation SA is equal to beta times A, and beta is--

OK. So this is something-- so it turns out-- and I don't think I have time to prove it, but it turns out you can prove it-- if I would have another three minutes-- you can prove it that the eigenvalues of matrix S-- eigenvectors, sorry, eigenvectors of matrix S are the same as eigenvectors of the full motion matrix.

So in other words, our motion matrix M minus 1 K-- this is the matrix. Then we have a matrix S. And normal modes are, you have a normal frequency and they have a shape. You have a normal vector, the ratio of amplitudes. And turns out that eigenvectors here, so the A's are the same. And again, I don't have time to show it, but you can show that this is the case. So if you have a symmetry in the system, then you can simply find eigenvectors of the thing to obtain the normal modes.

So if I look at my two pendula here, the symmetry is this way, so I have to have one which is fully symmetric, like this, and I have another one which is antisymmetric. Plus 1, minus 1, plus 1, minus 1. Similarly, here I have-- let's say if I have two masses, there is one motion which is like this, and one motion which is like that, because of the mirror symmetry. And you can show that if you have some other symmetries, like on a circle et cetera, that you have a similar type of fact. So you can build up on this symmetry argument. And finding eigenvectors of a matrix

0, minus 1, minus 1, 0 is infinitely simpler than finding matrix with G's and K's and everything, right?

All right. So thank you very much, and I hope this was educational.