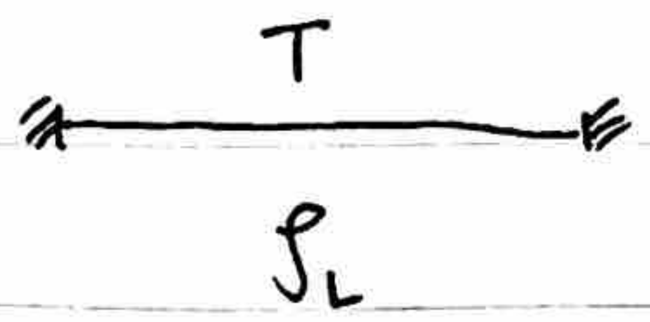
Systems we have learned :

Wave Equation :

$$\frac{\partial^2 \psi}{\partial t^2} = v_p^2 \frac{\partial^2 \psi}{\partial x^2}$$

There are three different kinds of systems discussed in the lecture :

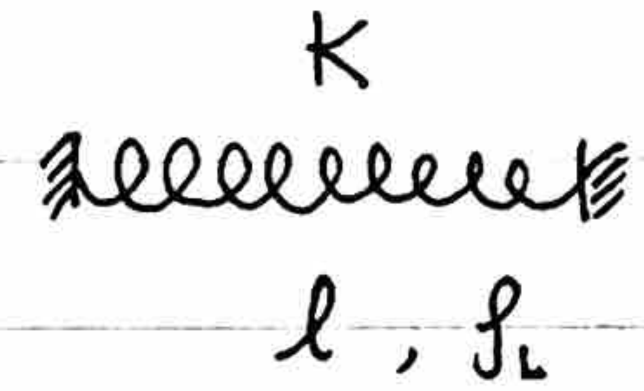
(1)



String with constant tension and mass/unit length ρ_L

$$v_p = \sqrt{\frac{T}{\rho_L}}$$

(2)

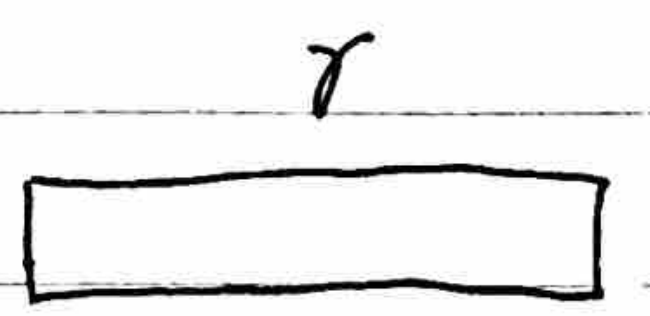


Spring with spring constant K, length l and mass/unit length ρ_L

$$v_p = \sqrt{\frac{Kl}{\rho_L}}$$

Georgi 7.1-2

(3)



Organ pipe with ~~room~~ pressure P_0 and air density ρ

$p \propto V^{-\gamma}$

$$v_p = \sqrt{\frac{\gamma P_0}{\rho}} .$$

Georgi 7.3

This time: EM waves !

Maxwell's Equations :

ε_0: permittivity

μ_0: permeability

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 & \Rightarrow \text{Gauss's Law} \\ \vec{\nabla}\cdot\vec{B} = 0 & \Rightarrow \text{Gauss's Law for magnetism} \\ \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} & \Rightarrow \text{Faraday's Law} \\ \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \underbrace{\varepsilon_0 \frac{\partial \vec{E}}{\partial t}}\right) & \Rightarrow \text{Ampere's Law} \end{cases}$$

No monopole "yet" :)

Displacement current

Maxwell's Addition !!

In vacuum : ($\rho = 0$, $\vec{J} = 0$)

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} & \leftarrow \text{Changing } \vec{B} \Rightarrow \vec{E} \\ \vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t} & \leftarrow \text{changing } \vec{E} \Rightarrow \vec{B} \end{cases}$$

Can you see the EM wave solution from the four equations ?

Maxwell saw it !!

We need to use this identity :

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \underbrace{(\vec{\nabla} \cdot \vec{\nabla})}_{\text{Laplace } \vec{\nabla}^2} A \quad — ①$$

In vacuum:

$$\underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{E})}_{= -\frac{\partial \vec{B}}{\partial t}} = \vec{\nabla}(\underbrace{\vec{\nabla} \cdot \vec{E}}_{=0}) - \vec{\nabla}^2 E$$

LHS

$$\Rightarrow \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

RHS

$$= -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = -\vec{\nabla}^2 E$$

$$\Rightarrow \vec{\nabla}^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\vec{E} + \frac{\partial^2}{\partial y^2}\vec{E} + \frac{\partial^2}{\partial z^2}\vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Wave Equation !!! (3D W.E)

This equation changed the world!

Maxwell is the first one who recognized it because of the term he put in.

It is a wave equation with speed

$$v_p = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \approx 3 \cdot 10^8 \, m/s \quad \text{Speed of light}$$

What about B field?

We can do the same exercise with B:

$$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

It is very important that the associated magnetic field also satisfies the wave equation.

From the Maxwell equation, $\vec{E}$ creates $\vec{B}$, $\vec{B}$ creates $\vec{E}$, therefore they can not exist without each other.

1638 Galileo : Speed of light is large
1676 Romer : 2.2×10^8 m/s
1729 James Bradley : 3.01×10^8 m/s

This means that in vacuum you can excite EM wave!

What is oscillating? Field!

EM waves!

Before that, lets review divergence and curl briefly

* Field:

Scalar field: Every position in the space gets a number.

ex: $T(x,y,z)$

Vector field. Instead of number or scalar you get a vector for each position

$\vec{A}(x,y,z) = \{A_x, A_y, A_z\}$
$= A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$

→ Electric and magnetic field: Vector field.
has direction

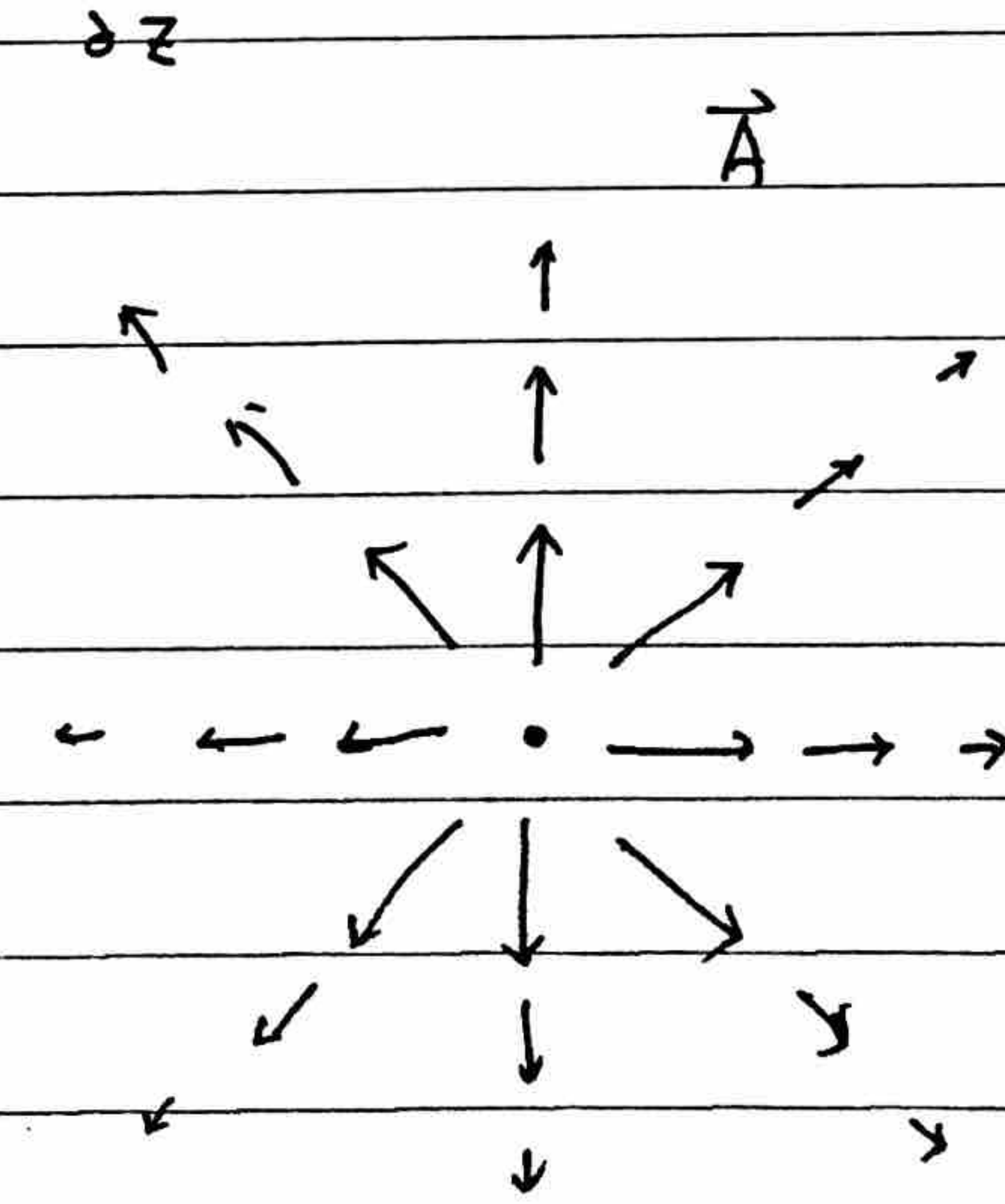
$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

To understand the structure of the vector fields:

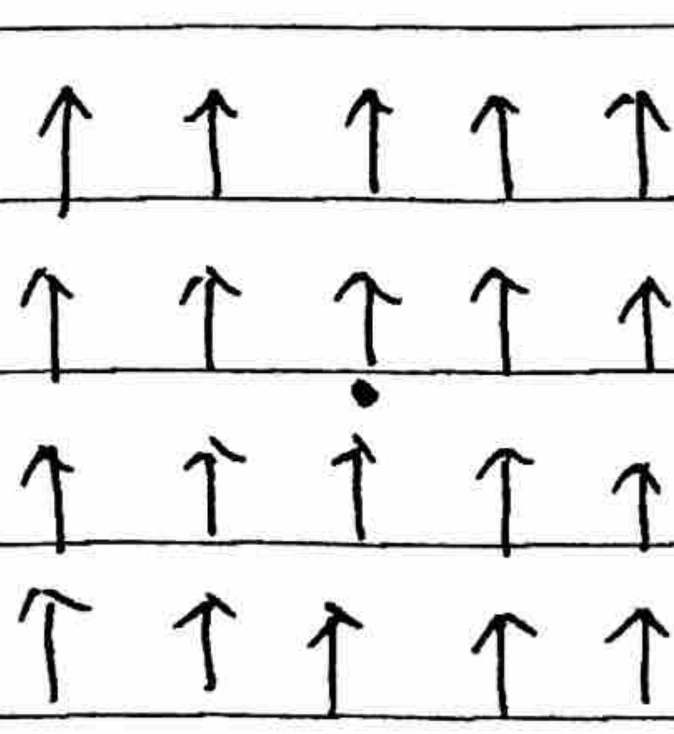
Divergence: $\vec{\nabla} = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

This is a measure of how much the vector v spreads out (Diverges) from the point in question.



Divergence of this vector field is $\boxed{\cancel{\text{negative}}}$. positive



Divergence = 0

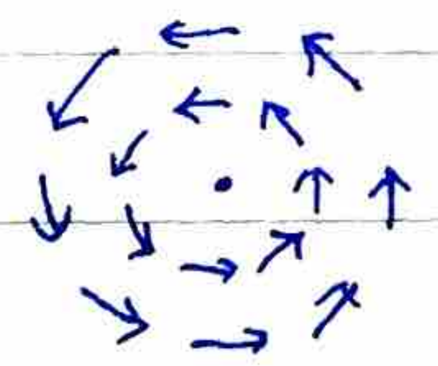
Curl: $\vec{\nabla} \times \vec{A}$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

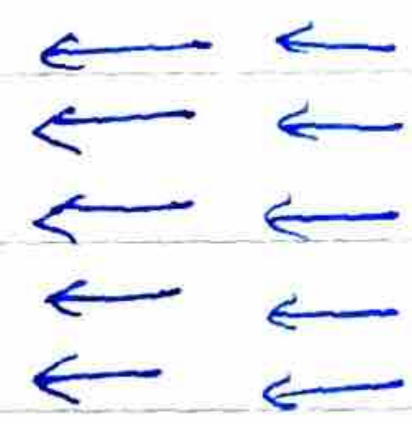
$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z}$$

What does "Curl" mean?

large curl

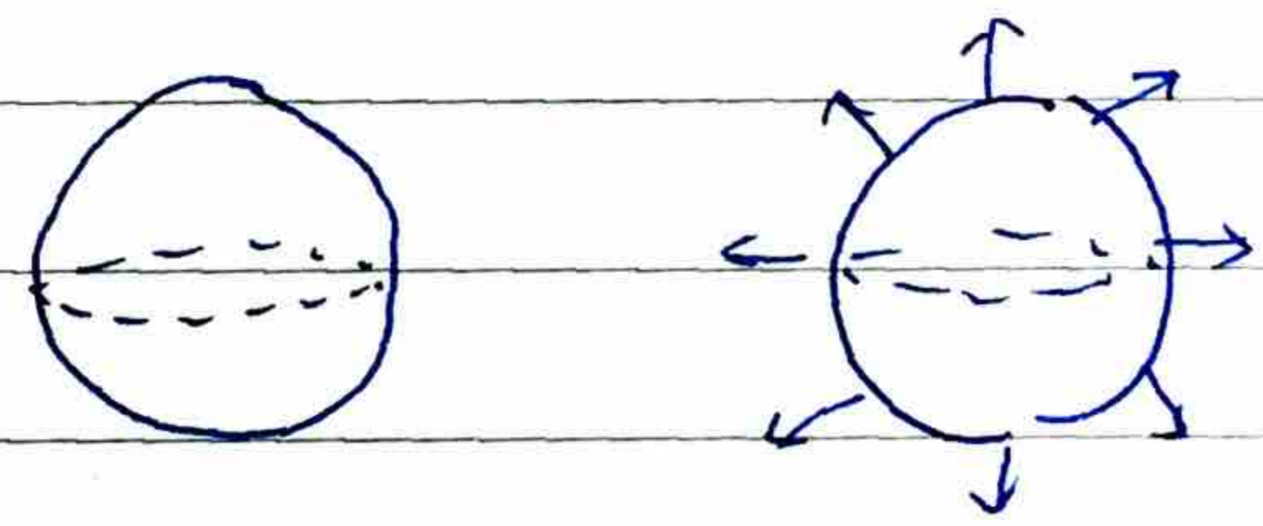


no curl.



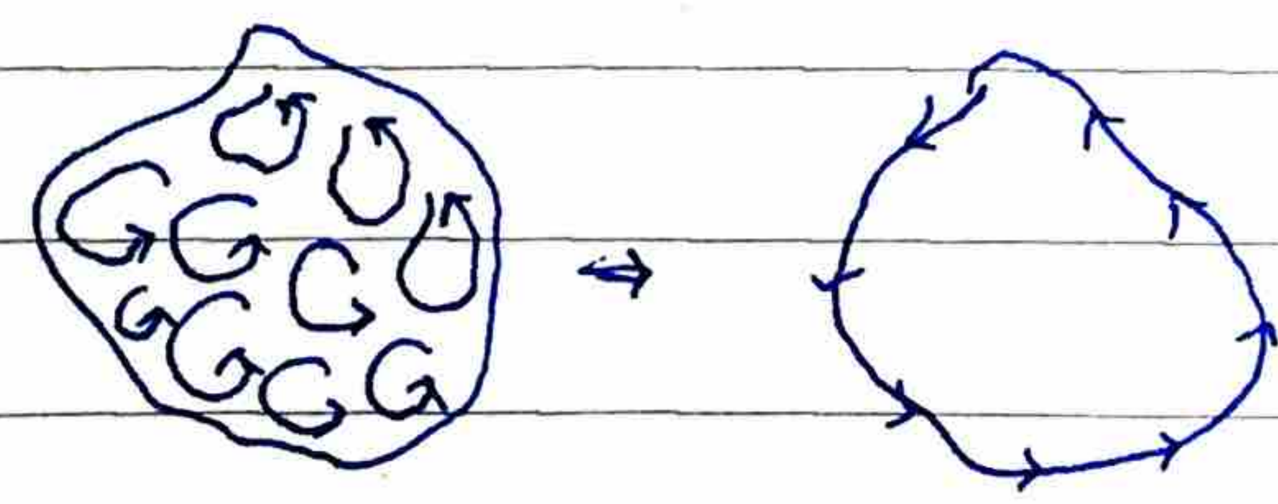
It is a measure of how much the vector $\vec{A}$ "curls around" the point of question

Gauss's Theorem



$$\int_V (\vec{\nabla} \cdot \vec{A})\, d\tau = \oint_S \vec{A} \cdot d\vec{a}$$

Stokes' Theorem



$$\int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \oint_P \vec{A} \times d\vec{l}$$

* Consider a "plane wave" solution

$$\vec{E} = \mathrm{Re}\left(E_0 e^{i(kz-\omega t)}\hat{x}\right)$$ Only in the $\hat{x}$ direction

$$= \{ E_0 \cos(kz-\omega t),\ 0,\ 0 \}$$

Check if it satisfies $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t^2}$

$$\Rightarrow \frac{\partial^2 E_x}{\partial z^2}\hat{x} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}\hat{x}$$

In $\hat{x}$ direction: $-E_0 k^2 \cos(kz-\omega t) = -\mu_0 \varepsilon_0 \omega^2 E_0 \cos(kz-\omega t)$

$$\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$$ $\Rightarrow$ condition needed to satisfy the wave eq.

* What about $\vec{B}$?

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \frac{\partial E_x}{\partial z}\hat{y} - \underbrace{\frac{\partial E_x}{\partial y}}_{=0}\hat{z} = -k E_0 \sin(kz-\omega t)\hat{y}$$

$$\Rightarrow \vec{B} = \frac{k}{\omega} E_0 \cos(kz-\omega t)\hat{y} = \frac{E_0}{c}\cos(kz-\omega t)\hat{y}$$

What do we learn from this exercise ?

(1) $\vec{E}$ must come with $\vec{B}$

In vacuum: $\vec{B} \perp \vec{E}$ and they are in phase

If $\vec{k}$ is the direction of propagation

$$\vec{B} = \frac{1}{c}\hat{k} \times \vec{E}$$

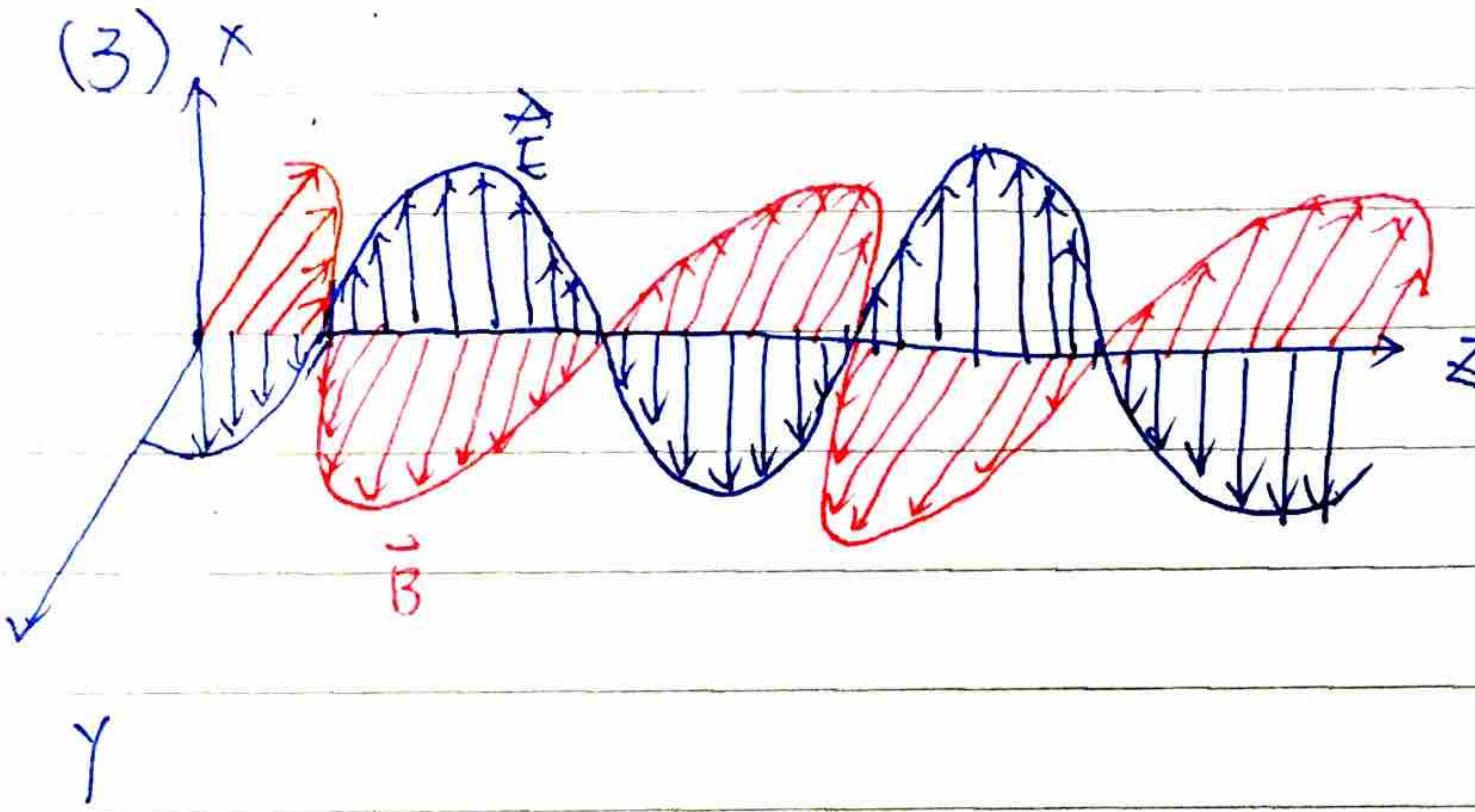
Amplitude : $E \Rightarrow E_0$

$B \Rightarrow \frac{E_0}{c}$

(2) The EM wave is non-dispersive C is independent of k

$$\left(\frac{\omega}{k} = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \right)$$

(3)



DEMO

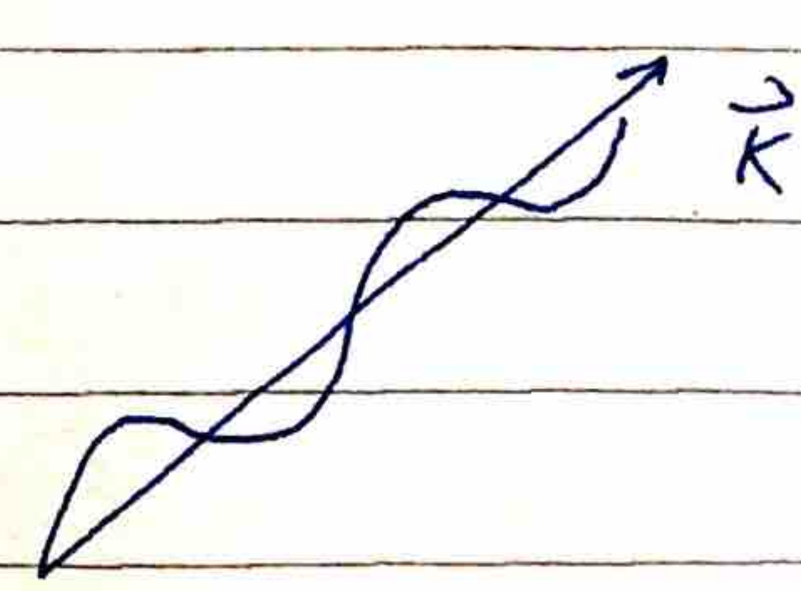
Direction of propagating EM wave : $\vec{E} \times \vec{B}$

In general a propagating EM wave can be written as:

$$\vec{E}(\vec{r}, t) = \mathrm{Re}\left(\vec{E}_0 \, e^{j(\vec{k}\cdot\vec{r} - \omega t \overset{\text{in general}}{+\phi})}\right)$$

$$\vec{E}_0 = E_{0x}\hat{x} + E_{0y}\hat{y} + E_{0z}\hat{z}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$



$$\omega = ck.$$

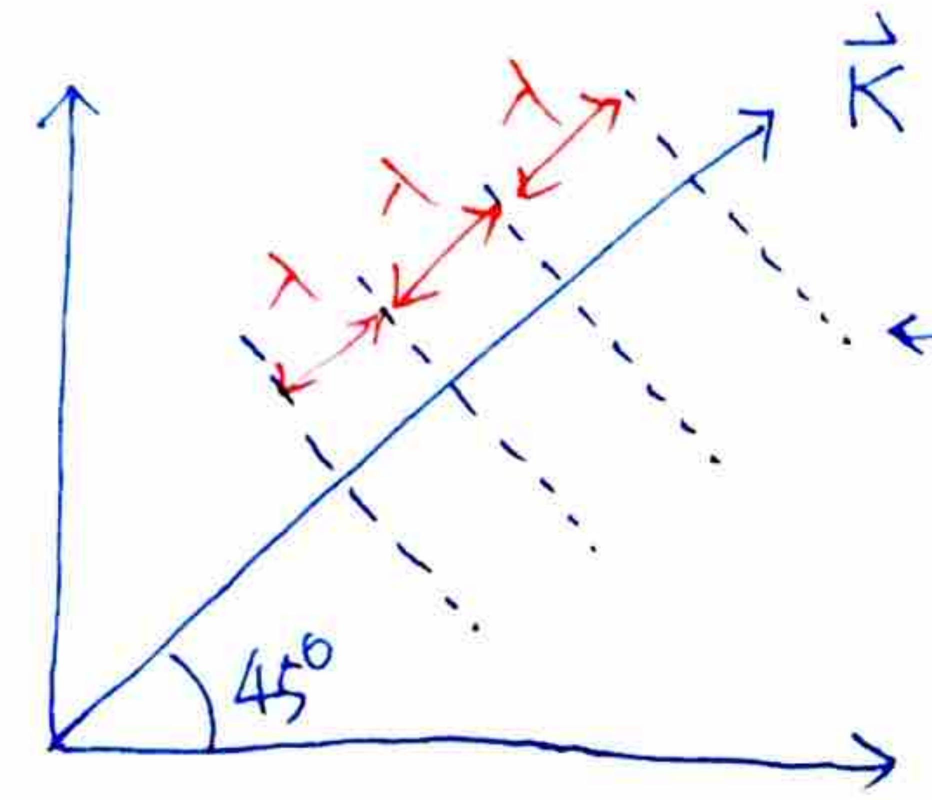
$$\vec{B}(\vec{r}, t) = \frac{1}{c}\hat{k} \times \vec{E}$$

For example: $\vec{k} = \frac{2\pi}{\lambda}\left[\frac{\hat{x}}{\sqrt{2}} + \frac{\hat{y}}{\sqrt{2}}\right]$

$$\vec{E_0} = \frac{-E_0}{\sqrt{2}}\hat{x} + \frac{E_0}{\sqrt{2}}\hat{y}$$

$$\vec{k}\cdot\vec{r} = \frac{2\pi}{\sqrt{2}\lambda}(x+y)$$

$$\Rightarrow \vec{E}(x,y,t) = E_0\left(-\frac{\hat{x}}{\sqrt{2}} + \frac{\hat{y}}{\sqrt{2}}\right)\cos\left(\frac{\sqrt{2}}{\lambda}\pi(x+y) - \omega t\right)$$

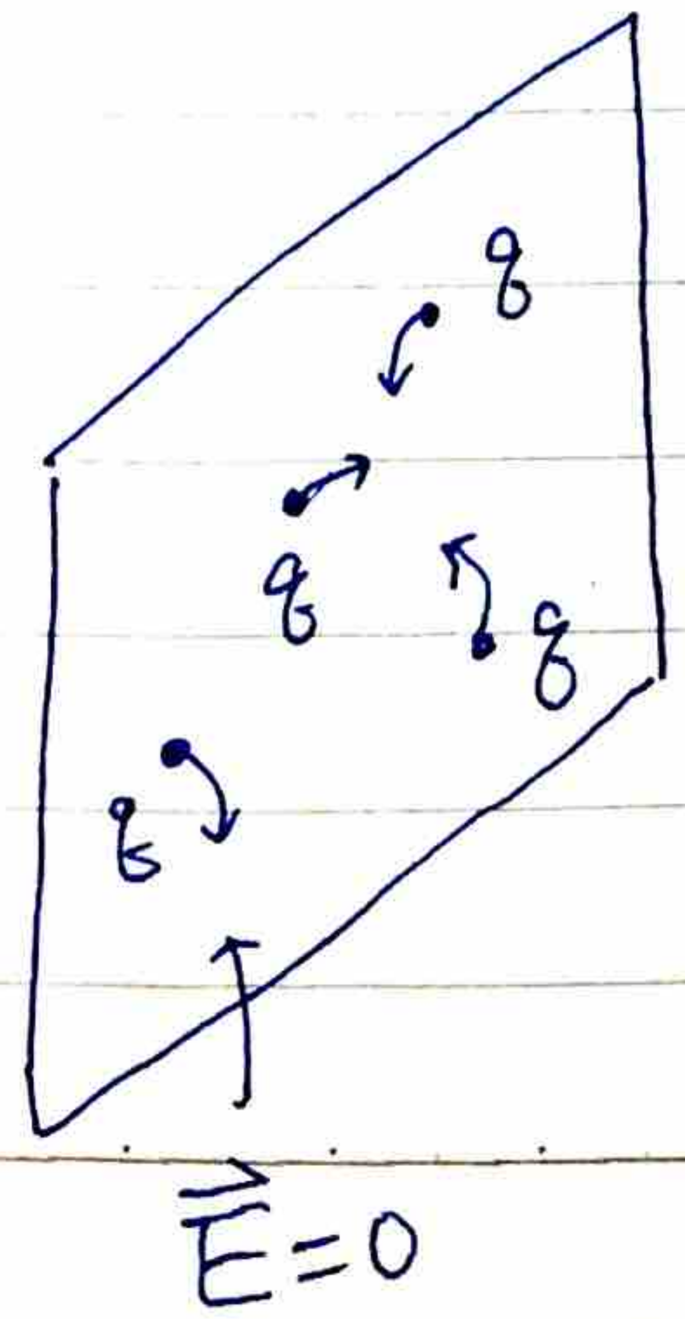


$$\vec{B} = \frac{1}{c}\hat{k}\times\vec{E} \quad \Rightarrow \vec{B}(x,y,t) = \frac{E_0}{c}\hat{z}\cos\left(\frac{\sqrt{2}}{\lambda}\pi(x+y) - \omega t\right)$$

If there is no other material, this EM wave will travel forever ....

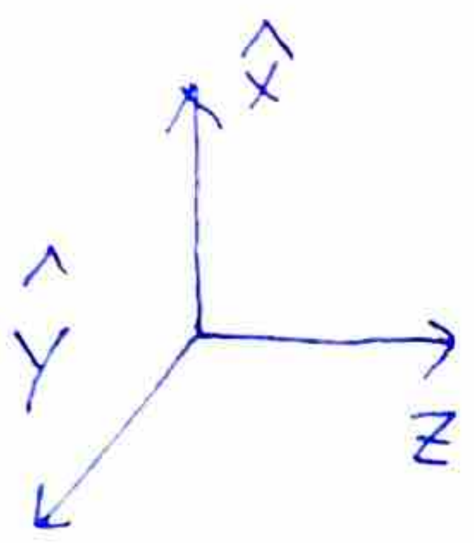
Now let's put something into the game :

A "perfect conductor"



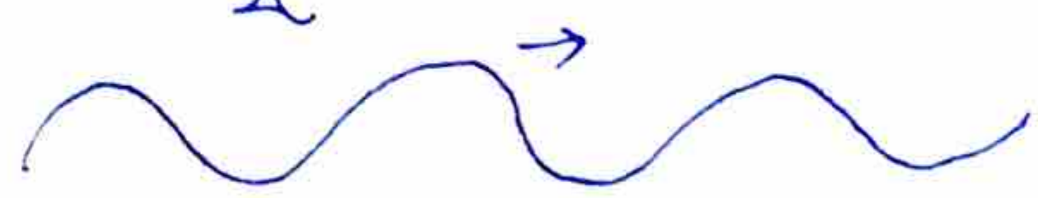
A busy world inside this system !

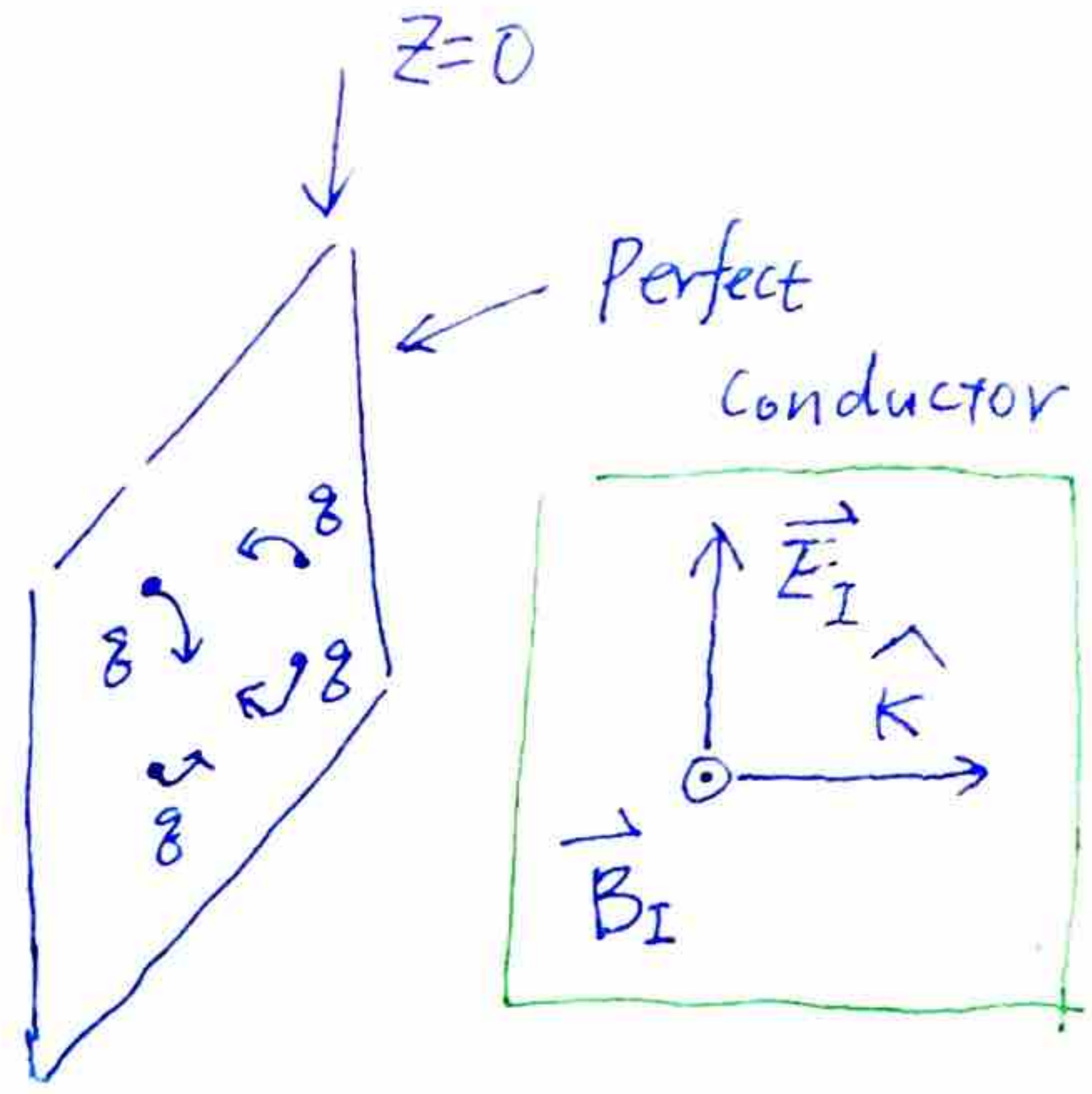
All the little charges are moving around without cost of energy (dissipation)



Incident Wave:

$$\begin{cases} \vec{E}_I = \dfrac{E_0}{2} \cos(kz - \omega t)\, \hat{x} \\ \vec{B}_I = \dfrac{E_0}{2c} \cos(kz - \omega t)\, \hat{y} \end{cases}$$





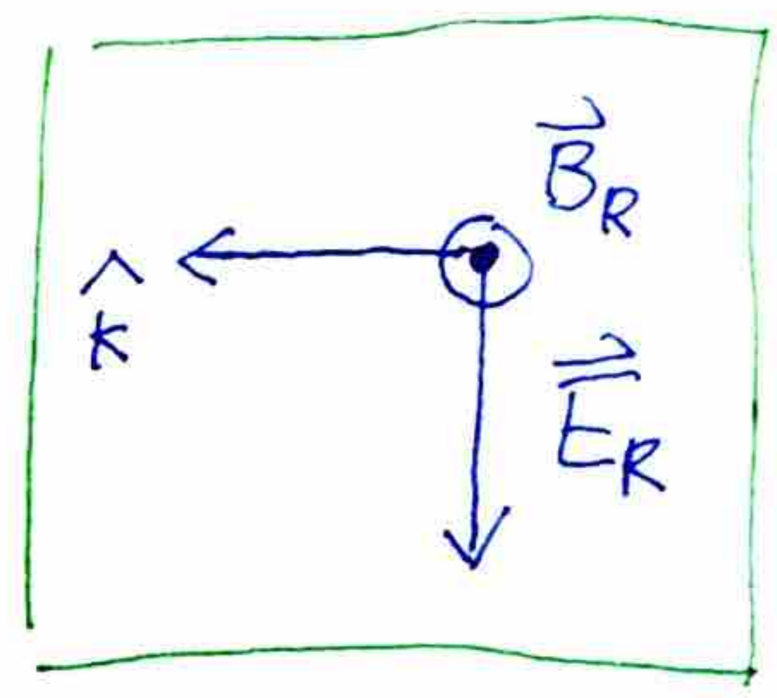
To satisfy the boundary condition
$\vec{E} = 0$ at $z = 0$

$\Rightarrow$ Need a reflected wave!

$$\vec{B}_R = \frac{1}{c} \hat{k} \times \vec{E}_R$$

$$\vec{E}_R = -\frac{E_0}{2} \cos(-kz - \omega t)\, \hat{x}$$

$$\vec{B}_R = \frac{E_0}{2c} \cos(-kz - \omega t)\, \hat{y}$$

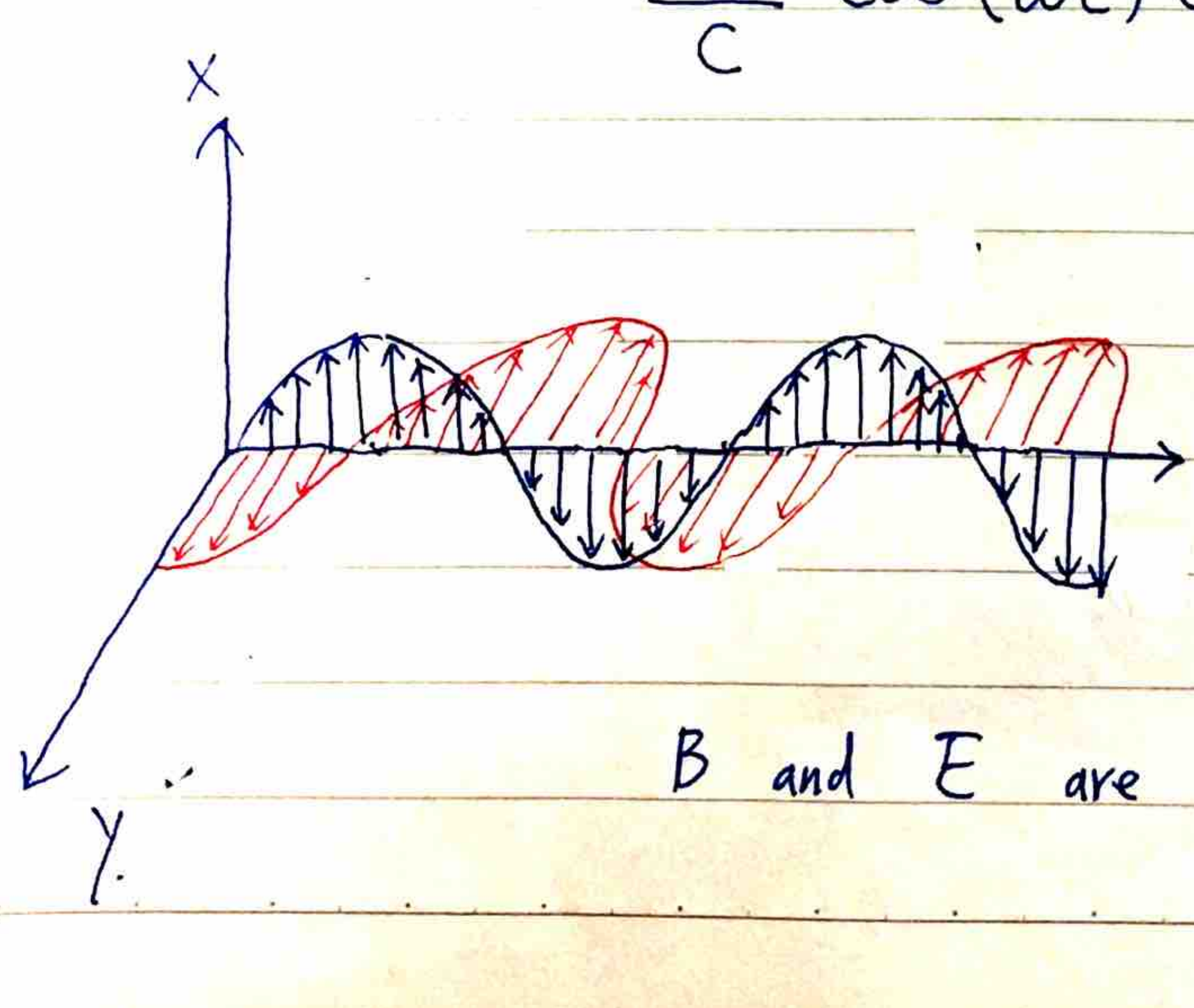


$$\Rightarrow \vec{E} = \vec{E}_I + \vec{E}_R = \frac{E_0}{2}\Big(\cos(kz - \omega t) - \cos(-kz - \omega t)\Big)\, \hat{x}$$

$$= +E_0 \sin(\omega t) \sin(kz)\, \hat{x}$$

$$\vec{B} = \vec{B}_I + \vec{B}_R = \frac{E_0}{2c}\Big(\cos(kz - \omega t) + \cos(-kz - \omega t)\Big)\, \hat{y}$$

$$= \frac{E_0}{c} \cos(\omega t) \cos(kz)\, \hat{y}$$



Standing Waves!!

B and E are not in phase!!!!

Energy density ?

$$U_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{\varepsilon_0}{2} E_0^2 \sin^2 \omega t \; \sin^2 kz$$

$$U_B = \frac{1}{2\mu_0} B^2 = \frac{\varepsilon_0}{2} E_0^2 \cos^2 \omega t \; \cos^2 kz$$

$\boxed{\text{Poynting vector}}$: directional energy flux or
the rate of energy transfer per unit area

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{1}{\mu_0} E_x B_y \hat{z}$$

$$= \frac{+E_0^2}{\mu_0 c} \sin(\omega t)\cos(\omega t)\cos(kz)\sin(kz) \quad \hat{z}$$

$$= \frac{+E_0^2}{4\mu_0 c} \sin(2\omega t)\sin(2kz)\;\hat{z}$$

This is how microwave owen works!

* The EM waves are bounced around inside the oven

* EM waves increase the vibration of the molecules in the oven and ~~product~~ increase the temperature of the food.