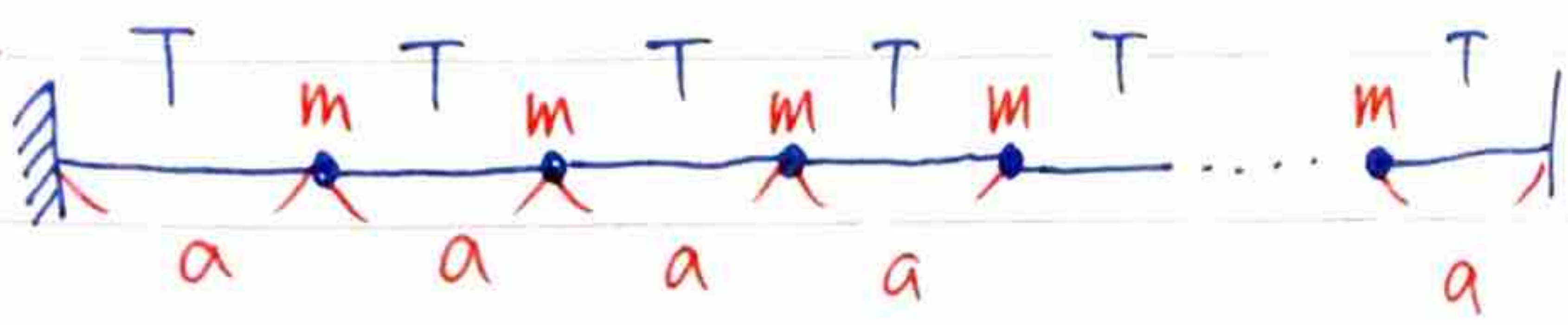
Exam 1

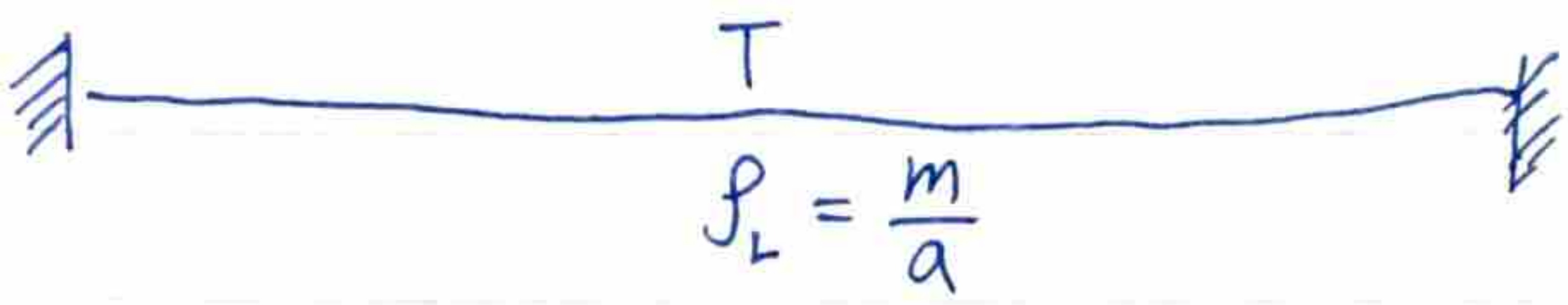
Last time:

System 1



$\downarrow$ $a \to 0$, $N \to \infty$ $\quad a \ll \frac{2\pi}{k} = \lambda$

System 2



$$-\ddot{X} = M^{-1}KX \quad ① \qquad\qquad M^{-1}KA = \omega^2 A$$

j_{th} term of $M^{-1}KA$: $\quad \omega^2 A_j = \frac{T}{ma}\left(-A_{j-1} + 2A_j - A_{j+1}\right)$

$\downarrow$ continuum limit

$$\omega^2 A(x) = \frac{T}{ma}\left(-A(x-a) + 2A(x) - A(x-a)\right)$$

Taylor Series

$$\approx \frac{T}{ma}\left(-\frac{\partial^2 A(x)}{\partial x^2} a^2\right)$$

$$= -\frac{T}{\rho_L}\frac{\partial^2 A(x)}{\partial x^2} \quad ②$$

$$\Rightarrow M^{-1}K \longrightarrow -\frac{T}{\rho_L}\frac{\partial^2}{\partial x^2} \qquad \psi_j \to \psi(x,t)$$

From ① and ② $\Rightarrow$ $\dfrac{\partial^2 \psi(x,t)}{\partial t^2} = \dfrac{T}{\rho_L}\dfrac{\partial^2 \psi(x,t)}{\partial x^2}$

Original Dispersion Relation

$$\omega^2 = 4\frac{T}{ma}\sin^2\frac{ka}{2}$$

$a \ll \frac{2\pi}{k} \Rightarrow ka$ very small

$$\Rightarrow \omega^2 \approx \frac{4T}{ma}\left(\frac{ka}{2}\right)^2 = \frac{T}{\rho_L}k^2$$

$$\Rightarrow v_p = \frac{\omega}{k} = \sqrt{\frac{T}{\rho_L}}$$

$$\Rightarrow \boxed{\frac{\partial^2\psi(x,t)}{\partial t^2} = v_p^2\frac{\partial^2\psi(x,t)}{\partial x^2}}$$

Wave Equation !!

= Infinite number of coupled equations of motion.

Come back to the original question 

What are the normal modes ?

$$\psi(x,t) = A(x)\, B(t)$$

Control the relative amplitude

control the time evolution

Plug into wave equation

$$A(x)\,\frac{\partial^2 B(t)}{\partial t^2} = v_p^2\, B(t)\,\frac{\partial^2 A(x)}{\partial x^2}$$

$$\frac{1}{v_p^2 B(t)}\,\frac{\partial^2 B(t)}{\partial t^2} = \frac{1}{A(x)}\,\frac{\partial^2 A(x)}{\partial x^2}$$

This Eq must be satisfied at all x and t !

$$\Rightarrow \frac{1}{v_p^2 B(t)}\,\frac{\partial^2 B(t)}{\partial t^2} = \frac{1}{A(x)}\,\frac{\partial^2 A(x)}{\partial x^2} = \underbrace{-K_m^2}_{\text{a constant}}$$

① $$\frac{1}{v_p^2 B(t)} \frac{\partial^2 B(t)}{\partial t^2} = -k_m^2$$

$$\frac{\partial^2 B(t)}{\partial t^2} = -k_m^2 v_p^2 B(t)$$

$$\Rightarrow B(t) = B_m \sin(\omega_m t + \beta_m)$$

$$\omega_m = v_p k_m$$

② $$\frac{1}{A(x)} \frac{\partial^2 A(x)}{\partial x^2} = -k_m^2$$

$$\Rightarrow A(t) = C_m \sin(k_m x + \alpha_m)$$

$$\Rightarrow \boxed{\psi_m(x,t) = A_m \sin(\omega_m t + \beta_m) \sin(k_m x + \alpha_m)}$$

m_{th} Normal mode.

$\omega_m = v_p k_m$ ⇒ decided by the property of the string

The two unknowns: α_m, k_m: decided by the boundary conditions

And A_m, β_m: decided by the initial conditions (will show that in the later discussion)

* Look at the structure of this normal mode solution:
Let's stop and think about what we have learned:

(1) Each point mass on the string is oscillating <u>harmonically</u> at the same frequency and phase!
only up and down, not in the horizontal direction!

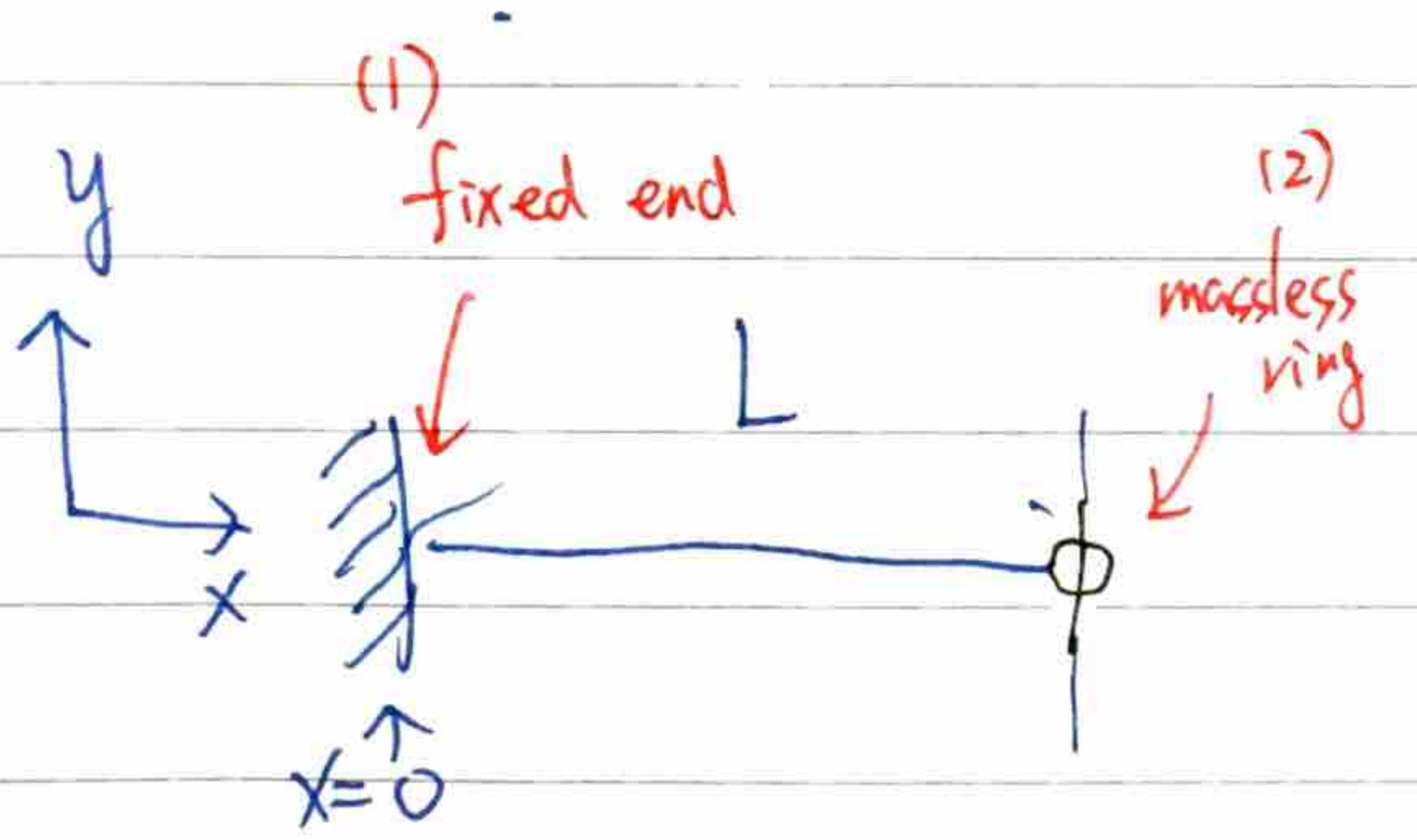
(2) Their relative amplitude = sin function!
↑ the same as discreet system

Need to determine unknown coefficients step by step.

* Let's take a concrete example:

Suppose we have a string, one end fixed
the other end: open

Slide 8 DEMO



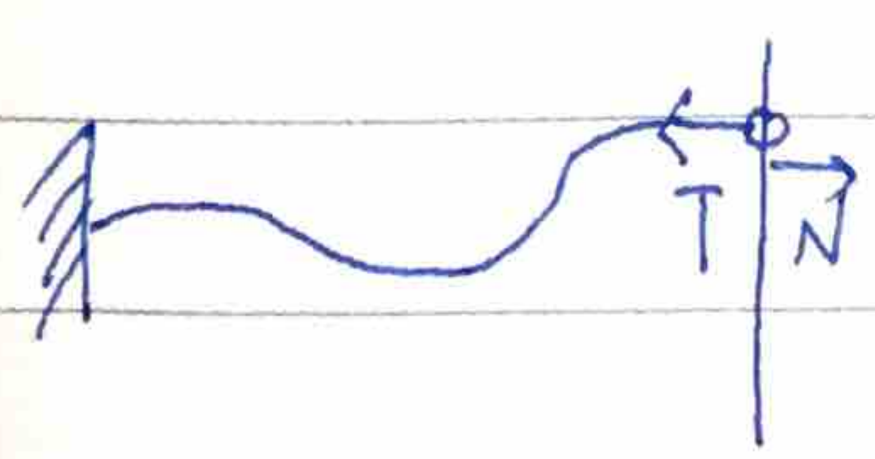
No friction.

Tension T, μ

Boundary condition:

(1) At $x=0 \Rightarrow \psi(0,t)=0$

(2) At $x=L \Rightarrow \frac{\partial \psi}{\partial x}(L,t)=0$



$T=N$

$a \to \infty$ ∵ the ring is massless!!
⇑

if $\frac{\partial \psi}{\partial x} \neq 0 \Rightarrow$ Net force! (T and N do not cancel!)

What are the normal modes? m_{th} mode

(1) $\Rightarrow$ $\psi_m(0,t) = A_m \sin(\alpha_m) \sin(\omega_m t + \beta_m)$

$= 0$

$\Rightarrow \alpha_m = 0$

(2) $\Rightarrow$ $\dfrac{\partial \psi_m}{\partial x} = A_m k_m \sin(\omega_m t + \beta_m) \cos(k_m x + \overset{0}{\overset{\shortparallel}{\alpha_m}})$

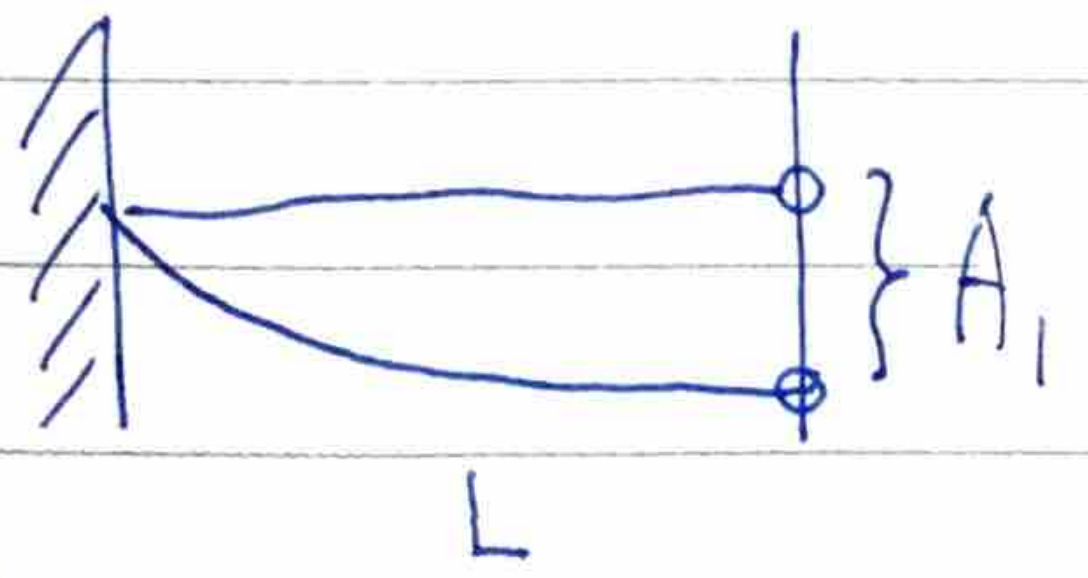
At $x=L$: $\dfrac{\partial \psi_m(L,t)}{\partial x} = 0 = A_m k_m \sin(\omega_m t + \beta_m) \cos(k_m L)$

$\Rightarrow k_m L = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5}{2}\pi, \ldots, \frac{(2m-1)}{2}\pi$

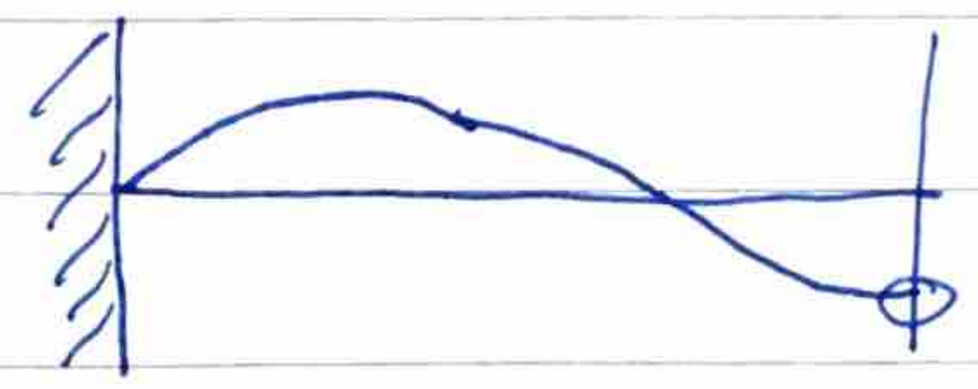
$k_m = \dfrac{(2m-1)\pi}{2L}$

$m=1$ $\quad k_1 = \dfrac{\pi}{2L}$ $\quad \lambda_1 = \dfrac{2\pi}{k_1} = 4L$

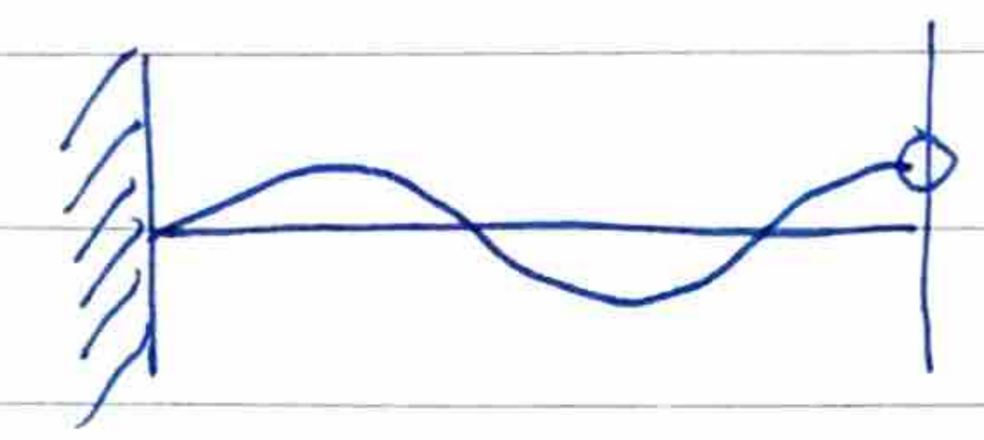
$\omega_1 = v k_1 = \sqrt{\dfrac{T}{\mu}} \dfrac{\pi}{2L}$



$m=2 \qquad k_2 = \dfrac{3\pi}{2L} \qquad \lambda_2 = \dfrac{4}{3}L$



$m=3 \qquad k_3 = \dfrac{5\pi}{2L} \qquad \lambda_3 = \dfrac{4L}{5}$



$\Rightarrow$ General Solution :

$$\psi(x,t) = \sum_{m=1}^{\infty} A_m \sin(\omega_m t + \beta_m) \cdot \sin(k_m x + \alpha_m)$$

From boundary conditions :

$$\alpha_m = 0 \,, \quad k_m = \frac{(2m-1)\pi}{2L}$$

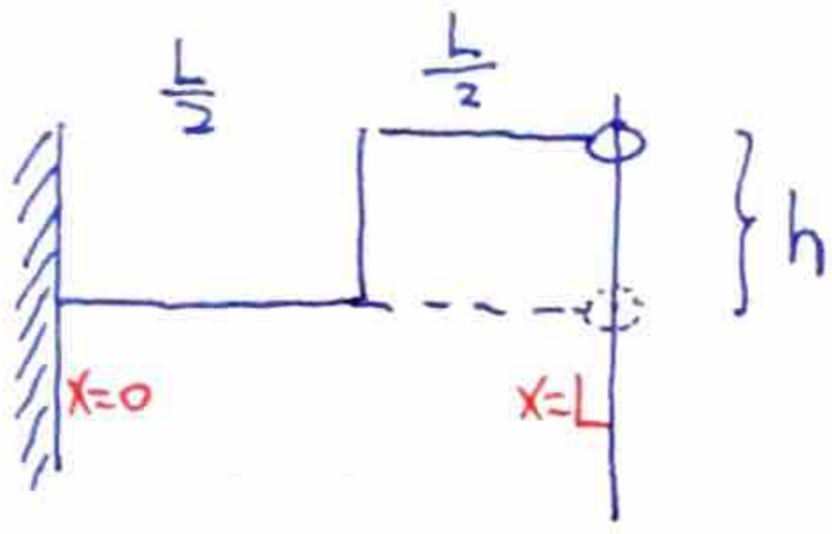
$$\psi(x,t) = \sum_{m=1}^{\infty} A_m \sin\left[\frac{(2m-1)v\pi}{2L} t + \beta_m\right] \sin\left[\frac{(2m-1)\pi}{2L} x\right]$$

Break?

$\sin(x)\sin(y) = \frac{1}{2}(\cos(x+y) + \cos(x-y))$

How do we extract A_m and β_m ?

$\Rightarrow$



Suppose at $t=0$, the string looks like this.

Also the string is at rest ($v(x)=0$)

$\Rightarrow$ Initial conditions: (a) $\dot{\psi}(x,0)=0$ (b) $\psi(x,0)$ is known.

From (a) we get $\dot{\psi}(x,t) = \sum_{m=1}^{\infty} A_m\omega_m \cos(\omega_m t + \beta_m)\sin(k_m x + \alpha_m)$

$\dot{\psi}(x,0) = 0 \Rightarrow \beta_m = \frac{\pi}{2} \Rightarrow \psi(x,0) = \sum_{m=1}^{\infty} A_m \sin(k_m x + \alpha_m)$

DEMO

$\alpha_m = 0$, $k_m = \frac{(2m-1)\pi}{2L}$

(b) How do I extract A_m from the given $\psi(x,0)$?

$\Rightarrow$ Use the "orthogonality" of sine functions

$$\int_0^L \sin(k_m x)\sin(k_n x)\,dx = \begin{cases} \frac{L}{2} & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}$$

$\Rightarrow$ We can extract A_m by:

$$A_m = \frac{2}{L}\int_0^L \psi(x,0)\sin(k_m x)\,dx$$

In this example:

$$A_m = \frac{2}{L}\int_{L/2}^{L} h \sin(k_m x)\,dx$$

$$= \frac{2}{L}\,\frac{-h}{k_m}\left[\cos(k_m L) - \cos\left(k_m \frac{L}{2}\right)\right]$$

where $k_m = \frac{(2m-1)\pi}{2L}$